MONOIDAL INFINITY CATEGORY OF COMPLEXES FROM TANNAKIAN VIEWPOINT

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ABSTRACT. In this paper we prove that a valued point into a scheme or stack naturally corresponds to a symmetric monoidal functor between stable ∞-categories of quasi-coherent complexes. It can be viewed as a derived analogue of Tannaka duality. As a consequence, we deduce that an algebraic stack satisfying a certain condition can be recovered from the stable ∞-category of quasi-coherent complexes with tensor operation.

1. INTRODUCTION

Let X be a reasonably nice scheme (for example, a noetherian scheme). Many important invariants of X come from the category of complexes of coherent sheaves on X. Practically speaking, by the category of complexes we mean the derived category of coherent complexes. The triangulated category equips some natural additional structures such as tensor structure arising from the derived tensor product, the canonical t-structure, etc. The symmetric monoidal (tensor) structure naturally determines the intersection products on algebraic K-theory groups, and thus this product yields the ring structures on various cohomology theories. Let us consider the tensor triangulated category $(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$ of perfect complexes on X, endowed with the derived tensor product $\otimes^{\mathbf{L}}$. In the remarkable papers [1], [2], Balmer proved that the tensor triangulated category $(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$ remembers the scheme X, that is to say, the whole scheme X can be recovered from tensor triangulated category $(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$. Balmer's reconstruction uses the classification of tensor thick subcategories of $\mathrm{D}_{\mathrm{perf}}(X)$, which has been studied by Hopkins [8], Neeman [24] and Thomason [32]. Roughly speaking, the reconstruction proceeds as follows. To the tensor triangulated category $(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$ he associates a ringed topological space which we shall denote by $\mathrm{Spec}(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$. A point on the topological space $\mathrm{Spec}(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$ corresponds to a tensor thick subcategory of $(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$ which satisfies a certain condition. Making use of Thomason's classification of tensor thick subcategories of $(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$ in terms of algebraic cycles, Balmer showed that the ringed space $\mathrm{Spec}(\mathrm{D}_{\mathrm{perf}}(X), \otimes^{\mathbf{L}})$ is isomorphic to the ringed space X.

We are motivated by this reconstruction problem and the classical Tannaka duality. Our principal idea is to view the reconstruction of a scheme from the tensor triangulated category of perfect complexes as a derived analogue of Tannaka duality. Let G be an affine group scheme over a field k. Then Tannaka duality states that G can be reconstructed from the tensor abelian category of finite dimensional representations of G. More precisely, if S is an affine k-scheme and $\mathrm{Hom}_k(S, BG)$ is the groupoid of S-valued points (over k), then Tannaka duality gives an equivalence

$$\mathrm{Hom}_k(S, BG) \longrightarrow \mathrm{Fun}_k(\mathrm{VB}^{\otimes}(BG), \mathrm{VB}^{\otimes}(S)) : \ f \mapsto f^*$$

where BG is the classifying stack of G, $\mathrm{VB}^{\otimes}(\bullet)$ denotes the tensor exact category of vector bundles, and $\mathrm{Fun}_k(\mathrm{VB}^{\otimes}(BG), \mathrm{VB}^{\otimes}(S))$ is the category of tensor exact k-linear functors. (See [7], [29] for the precise statement.)

Let X be a scheme or algebraic stack (satisfying a certain condition) and let $\mathrm{D}_{\mathrm{perf}}(X)$ be the triangulated category of perfect complexes on X. Our main goal is (roughly speaking) to establish a derived analogue of Tannaka duality Theorem 5.9, which relates the category of morphisms $S \to X$ from a scheme S to an algebraic stack X with the category of exact functors $\mathrm{D}_{\mathrm{qcoh}}(X) \to \mathrm{D}_{\mathrm{qcoh}}(S)$ that preserves derived tensor products. Besides the appealing Tannakian viewpoint our approach has the virtue of recovering rich data. Thick subcategories and tensor thick subcategories of a (tensor) triangulated category give rise to localizations. For a (nice) scheme X, localizations of the triangulated category $\mathrm{D}_{\mathrm{perf}}(X)$ arising from Zariski open sets are described in terms of tensor thick subcategories, and it enables one to reconstruct X. However, if a (tensor) triangulated category D is the derived category arising from algebraic stacks (including the derived category of complexes of representations of an algebraic group) and representations of quivers, the data of tensor thick subcategories in D is not enough to recover the original sources such as stacks and quivers, and they happen to be trivial. For instance, if X is a Deligne-Mumford stack satisfying a certain condition, the recent result of Krishna [15] shows that only the coarse moduli space M for X can be recovered from the data of tensor thick subcategories in $\mathrm{D}_{\mathrm{perf}}(X)$. In our Tannakian approach, we treat the data arising from symmetric monoidal functors which are not necessarily localizations. As a consequence, our reconstruction is applicable to a fairly large class of Deligne-Mumford stacks. The stabilizer group at a point on a stack is described as the automorphism group of monoidal natural transformations.

One noteworthy feature of our approach is the usage of higher category theory. The natural machinery of higher categories allows us to formulate and study our derived Tannaka duality Theorem 5.9. In addition, it enables us to prove a categorical characterization of derived functors associated to morphisms of schemes and stacks (Theorem 5.13). In order to treat symmetric monoidal functors and realize the derived Tannaka formalism we shall replace the triangulated category $\mathrm{D}_{\mathrm{perf}}(X)$ by "enhanced" (higher) category $\mathcal{D}_{\mathrm{perf}}(X)$. There are some candidates which provide the frameworks dealing with such enhanced higher categories: triangulated derivators, dg-categories, stable simplicial categories, stable Segal categories and stable ∞-categories (quasi-categories), etc. We use the theory of ∞-categories (quasi-categories) which has been extensively developed by Joyal and Lurie [13], [18]. In addition, many parts of this paper depend on the theory of ∞-categories and theorems such as derived Morita theory [34], [4] build on the higher category theory.

This paper is organized as follows. In Section 2, we begin by reviewing the basic notions on ∞-categories in the sense of [13] and [18] and we give preliminaries related to our study. In Section 3, we prove some lemmas concerning Kan extensions in the ∞-categorical setting, which we will use later. In Section 4, we then proceed to study some property of symmetric monoidal functors between stable symmetric monoidal ∞-categories of quasi-coherent complexes. In our study derived Morita theory plays an important role. In Section 5, applying results of Section 3 and 4, we will prove main results of this paper.

2. Preliminaries and ∞-category of complexes

In this section, we will fix notion and convention and prepare the settings. We begin by reviewing the theory of ∞-categories which we will use in the course of this paper. Roughly speaking, an $(\infty, 1)$-category or simply an ∞-category is a weak ∞-category whose n-morphisms are invertible for $n > 1$. At present, there are at least four approaches to such a theory: simplicial categories, Segal categories, complete Segal spaces and quasi-categories. It is known that all four theories are equivalent. In other words, each theory is linked to one another via a Quillen equivalence (see [14], [5]). Among them, we use the theory of quasi-categories ([12], [13], [18]), which we shall call ∞-categories. We review basic definitions and facts on quasi-categories for the convenience of the reader. However, it is an almost impossible task to present a rapid overview of all materials [12], [13], [18], [21], [22], [23] and thus our review is a quick introduction to basic notions on quasi-categories, which are appearing in the first Chapter of [18]. Therefore we refer to the book [18] as the general reference of the theory of quasi-categories.

2.1. ∞-**categories.** Let us recall the definition of a quasi-category. A (small) *quasi-category* S is a (small) simplicial set such that for any $0 < i < n$ and any diagram

$$
\begin{array}{ccc}
\Lambda_i^n & \longrightarrow & S \\
\downarrow & \nearrow & \\
\Delta^n & &
\end{array}
$$

of solid arrows, there exists a dotted arrow filling the diagram. Here Λ_i^n is the i-th horn and Δ^n is the standard n-simplex. Following [18], in the sequel we call quasi-categories ∞-*categories*. A functor of ∞-categories $S \to S'$ is a map of simplicial sets. By the definition, ∞-categories form a full subcategory of simplicial sets. It contains Kan complexes. The ∞-categories also generalize (nerves of) ordinary categories (cf. [18, 1.1.2]). Let Δ^1 be the standard 1-simplex. It can be regarded as the nerve of the category $\{0, 1\}$ which consists of two objects 0, 1, and the nondegenerate morphism $0 \to 1$. Similarly, Δ^0 can be considered to be the category having one object with the identity. Let S be the nerve of the category $\{0 \overset{a,b}{\rightleftarrows} 1\}$ such that $a \circ b = \mathrm{Id}, b \circ a = \mathrm{Id}$. Three simplicial sets Δ^1, Δ^0 and S are all *weak homotopy equivalent* to one another, and they are *not isomorphic* to one another as simplicial sets. However, from the viewpoint of category theory, we should consider that Δ^0 is "equivalent" to S, and Δ^1 is not "equivalent" to Δ^0 and S. Hence it is necessary to have a correct notion of equivalences which generalizes the notion of equivalences of ordinary categories. The important concept we first recall is *categorical equivalences* between simplicial sets. Let Set_Δ be the category of simplicial sets and let Cat_Δ be the category of simplicial categories, in which morphisms are simplicial functors. Here a simplicial category is a category enriched over the category of simplicial sets. Let $\mathcal{H}$ be the homotopy category of "spaces", that is, the category obtained from Set_Δ by inverting weak homotopy equivalences. To a simplicial category $\mathcal{C}$, applying $\mathrm{Set}_\Delta \to \mathcal{H}$ to the mapping complexes in $\mathcal{C}$ we associate an $\mathcal{H}$-enriched category $\mathrm{h}\mathcal{C}$. Let $\mathcal{C}$ and $\mathcal{D}$ be two simplicial categories. A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an (*Dwyer-Kan*) *equivalence* (resp. *essentially surjective*) if the induced functor $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an

equivalence of $\mathcal{H}$-enriched categories (resp. essentially surjective). A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is fully faithful if, for any two objects $C, C' \in \mathcal{C}$ the induced morphism $\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{D}}(F(C), F(C'))$ is a weak homotopy equivalence. There is an adjoint pair [18, 1.1.5]

$$\mathfrak{C} : \mathrm{Set}_{\Delta} \leftrightarrows \mathrm{Cat}_{\Delta} : \mathrm{N}.$$

In this paper we will not use the detailed constructions of this adjunction and refer to [18, 1.1] for the definition of $\mathfrak{C}$ and N, but an important point is that the pair is a Quillen equivalence with respect to suitable model structures (see below). The functor N is called the simplicial nerve functor. In fact, if $\mathcal{C}$ is an ordinary category regarded as a simplicial category, then N($\mathcal{C}$) coincides with the usual nerve, and thus the simplicial nerve functor generalizes the classical nerve functor to the ∞-categorical setting. A map of simplicial sets $F : S \to T$ is a *categorical equivalence* (resp. *essentially surjective, fully faithful*) if $\mathfrak{C}(F) : \mathfrak{C}(S) \to \mathfrak{C}(T)$ is an equivalence (resp. essentially surjective, fully faithful) of simplicial categories.

For a simplicial set S we define h(S) to be h($\mathfrak{C}(S)$). Here we ignore the $\mathcal{H}$-enrichment of h($\mathfrak{C}(S)$) and refer to h(S) as a homotopy category of S. Here we will describe an alternative construction of a homotopy category of S when S is an ∞-category [18, 1.2.3]. Let $\pi(S)$ be the category defined in the following way. The objects of $\pi(S)$ are the vertices of S. For $f : \Delta^1 \to S$, $f(\{0\})$ and $f(\{1\})$ are said to be the source and the target, respectively. Let $s, s' \in S$ be two objects and let $f, g : \Delta^1 \rightrightarrows S$ be edges. Suppose that f and g have the same source s and target s'. Then f and g is said to be homotopic if there exists $\Delta^2 \to S$ determined by

$$
\begin{array}{ccc}
s & \xrightarrow{\ f\ } & s' \\
 & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \mathrm{Id}} \\
 & & s'.
\end{array}
$$

Then the relation of homotopy is an equivalence relation on edges from s to s'. Let $\mathrm{Hom}_{\pi(S)}(s, s')$ be the set of homotopy classes of edges joining s to s'. Using the definition of ∞-categories, we can define a composition law on the homotopy classes of edges. This yields a category $\pi(S)$ which turns out to be equivalent to h($\mathfrak{C}(S)$). Abusing notation we often write h(S) for $\pi(S)$.

The pair of adjoint ($\mathfrak{C}$, N) plays an important role in the various constructions of ∞-categories and their functors and so on. For various applications, it is better to view this adjoint as a Quillen adjoint pair with respect to suitable model structures on Set_{Δ} and Cat_{Δ} rather than a usual adjoint pair. The category Set_{Δ} admits a model structure, in which a weak equivalence is a categorical equivalence, and a cofibration is a monomorphism ([13], [18, 2.2.5.1]). It turns out that an object is fibrant in this model category if and only if it is an ∞-category. We refer to this model structure as *Joyal model structure*. There exists a model structure on Cat_{Δ} such that the weak equivalences are equivalences and fibrant objects are simplicial categories whose mapping complexes are Kan complexes (see for the details [5], [18, A 3.2]). Then the adjunction ($\mathfrak{C}$, N) is a Quillen equivalence with respect to these model structures (see [18, 2.2.5.1]). For example, we can use this adjoint as follows. Let M be a simplicial model category. Then the full subcategory M° spanned by cofibrant-fibrant objects is a fibrant simplicial category. (For a model category M we shall denote by M° the

full subcategory spanned by cofibrant-fibrant objects.) Applying the simplicial nerve functor to M° we obtain an ∞-category $\mathrm{N}(\mathsf{M}^\circ)$.

The Joyal model structure on Set_Δ is relevant to the usual model structure introduced by Quillen ([28]), in which a weak equivalence is a weak homotopy equivalence. According to [13] [18, 2.2.5.8], we have the following implications

$$(\text{isomorphisms}) \Rightarrow (\text{categorical equivalences}) \Rightarrow (\text{weak homotopy equivaleces}).$$

Moreover, there exists a left Bousfield localization

$$\mathrm{Set}_\Delta^{\mathrm{J}} \rightleftarrows \mathrm{Set}_\Delta^{\mathrm{Q}}$$

where $\mathrm{Set}_\Delta^{\mathrm{J}}$ (resp. $\mathrm{Set}_\Delta^{\mathrm{Q}}$) denotes the category Set_Δ equipped with Joyal model structure (resp. the usual model structure).

Let S be a simplicial set. An *object* in S is a vertex $\Delta^0 \to S$. A *morphism* in S is an edge $\Delta^1 \to S$, and when S is an ∞-category, a morphism $\Delta^1 \to S$ is said to be an *equivalence* if it gives rise to an isomorphism in the homotopy category $\mathrm{h}S$. Let $\mathcal{C}$ and $\mathcal{D}$ be two ∞-categories. Define $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to be the simplicial set $\mathrm{Map}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathcal{D})$ which parametrizes maps from $\mathcal{C}$ to $\mathcal{D}$, that is, a map $\Delta^n \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ amounts to a map $\mathcal{C} \times \Delta^n \to \mathcal{D}$. By [18, 1.2.7.3], the simplicial set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an ∞-category. We shall refer to an object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as a *functor* from $\mathcal{C}$ to $\mathcal{D}$. We shall refer to a *morphism* (resp. an *equivalence*, i.e., a *morphism* which induces an isomorphism in $\mathrm{h}\,\mathrm{Fun}(\mathcal{C}, \mathcal{D})$) in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as a *natural transformation* (resp. a *natural equivalence*). We define $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ to be the largest Kan complex of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Namely, $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ is the subcategory spanned by natural equivalences. Define $\mathcal{C}\mathrm{at}_\infty^\Delta$ to be a fibrant simplicial category whose objects are small ∞-categories, and whose hom simplicial set (between $\mathcal{C}$ and $\mathcal{D}$) is $\mathrm{Map}(\mathcal{C}, \mathcal{D})$. Let $\mathcal{C}\mathrm{at}_\infty$ be the simplicial nerve of $\mathcal{C}\mathrm{at}_\infty^\Delta$ (cf. [18, Chapter 3]). We shall refer to $\mathcal{C}\mathrm{at}_\infty$ as the ∞-category of (small) ∞-categories. We shall denote by $\widehat{\mathcal{C}\mathrm{at}}_\infty$ the ∞-category of (large) ∞-categories.

Let S be an ∞-category and let s, s' be two objects in S. In the course of the paper we sometimes discuss "the mapping space" from s to s'. The direct way to the definition is to define $\mathrm{Map}_S(s, s')$ to be the complex $\mathrm{Map}_{\mathfrak{C}(S)}(s, s')$. Remembering the relationship between ∞-categories and simplicial categories, we should regard the simplicial set $\mathrm{Map}_S(s, s')$ as an object in the homotopy category $\mathcal{H}$ of spaces. The simplicial set $\mathrm{Map}_{\mathfrak{C}(S)}(s, s')$ equips associative compositions (varying s and s'), but it is not a Kan complex in general. There are several ways to construct a simplicial set that represents the weak homotopy type of $\mathrm{Map}_{\mathfrak{C}(S)}(s, s')$. For example, a Kan complex of left morphisms $\mathrm{Hom}_S^{\mathrm{L}}(s, s')$ determined by

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{Hom}_S^{\mathrm{L}}(s, s')) = \{f : \Delta^{n+1} \to S \mid f|_{\Delta^{\{0\}}} = s,\ f|_{\Delta^{\{1,\dots,n+1\}}} \text{ is constant at } s'\}$$

represents the weak homotopy type of $\mathrm{Map}_{\mathfrak{C}(S)}(s, s')$. (see [18, 1.2.2] for details of mapping complexes).

2.2. ∞-category of quasi-coherent complexes.

We refer to [16] as the general reference of the notion of algebraic stacks. In this paper, all algebraic stacks (and schemes) are assumed to be *Deligne-Mumford*, *quasi-compact* and to *have affine diagonal*. We fix three Grothendieck universes $\mathbb{U}_1 \in \mathbb{U}_2 \in \mathbb{U}_3$ such that $\mathbb{U}_1$ contains all finite ordinals. We suppose that all schemes, rings and others belong to $\mathbb{U}_1$ and all (pre)sheaves are $\mathbb{U}_1$-small. Entries in $\mathbb{U}_1$ (resp. $\mathbb{U}_2$, $\mathbb{U}_3$) are called small (resp. large, super-large). By

a vector bundle on an algebraic stack $\mathcal{X}$ we mean a locally free $\mathcal{O}_{\mathcal{X}}$-module of finite type.

In the rest of this section, R is a commutative ring. We denote by $\mathrm{C}^{\bullet}(R)$ the category of cochain complexes of R-modules. Let $\mathrm{C}_{\bullet}(R)$ be the category of chain complexes of R-modules and let $\mathrm{C}_{\bullet \geq 0}(R)$ be the full subcategory of $\mathrm{C}_{\bullet}(R)$ consisting those objects $M_{\bullet}$ such that $M_n = 0$ for any $n < 0$. Let $\mathrm{C}^{\bullet \leq 0}(R)$ be the full subcategory of $\mathrm{C}^{\bullet}(R)$ consisting those objects $M^{\bullet}$ such that $M^n = 0$ for any $n > 0$. The standard truncation functor $\tau^{\leq 0} : \mathrm{C}^{\bullet}(R) \to \mathrm{C}^{\bullet \leq 0}$ is determined by $\tau^{\leq 0}(M^{\bullet})^p = 0$ if $p \geq 1$, $\tau^{\leq 0}(M^{\bullet})^0 = \ker(d_M^0)$ and $\tau^{\leq 0}(M^{\bullet})^p = M^p$ if otherwise. Here $d_M^0 : M^0 \to M^1$ is the 0-th differential of $M^{\bullet}$.

Let $S = \operatorname{Spec} A$ be an affine scheme. A quasi-coherent $\mathcal{O}_S$-complex is a set of data $\{M_B, \alpha\}_{\operatorname{Spec} B \to \operatorname{Spec} A}$ consisting of an (unbounded) complex M_B of B-modules for any $\operatorname{Spec} B \to \operatorname{Spec} A$, and an isomorphism $\alpha_{\phi} : M_B \otimes_B B' \to M_{B'}$ for any $\phi : \operatorname{Spec} B' \to \operatorname{Spec} B$ over $\operatorname{Spec} A$, such that α_{ϕ}'s satisfy the cocycle condition, i.e., for each $\phi \circ \psi : \operatorname{Spec} B'' \to \operatorname{Spec} B' \to \operatorname{Spec} B$ over $\operatorname{Spec} A$ we have $\alpha_{\phi \circ \psi} = \alpha_{\psi} \circ \alpha_{\phi}$. Let $\mathrm{QC}(S)$ be the category of quasi-coherent $\mathcal{O}_S$-complexes. Let $A\text{-Mod}_{\Delta}$ be the category of simplicial A-modules. For two complexes $F^{\bullet}, G^{\bullet}$ in $\mathrm{QC}(S)$, the mapping simplicial set is given by

$$\operatorname{Map}_{\mathrm{QC}(S)}(F^{\bullet}, G^{\bullet}) := \mathrm{K}(\tau^{\leq 0} \operatorname{Hom}_{\mathrm{QC}(S)}^{\bullet}(F^{\bullet}, G^{\bullet}))$$

where $\operatorname{Hom}_{\mathrm{QC}(S)}^{\bullet}(F^{\bullet}, G^{\bullet})$ denotes the internal Hom complex and K is the composition of the natural equivalence $\mathrm{C}^{\bullet \leq 0}(A) \xrightarrow{\sim} \mathrm{C}_{\bullet \geq 0}(A)$ and the Dold-Kan correspondence $\mathrm{C}_{\bullet \geq 0}(A) \xrightarrow{\sim} A\text{-Mod}_{\Delta}$ (here we regard the left-hand side as the underlying simplicial set). Each mapping simplicial set is a simplicial abelian group and thus $\mathrm{QC}(S)$ is a fibrant simplicial category. By the projective model structure [9, Section 4.2], $\mathrm{QC}(S)$ forms a symmetric monoidal model category, in which the weak equivalences are quasi-isomorphisms, and fibrations are termwise surjections. Let $\mathrm{QC}(S)^{\circ} \subset \mathrm{QC}(S)$ be the full subcategory spanned by cofibrant-fibrant objects. We define the stable ∞-category $\mathcal{D}_{\mathrm{qcoh}}(S)$ of quasi-coherent complexes to be the simplicial nerve $\mathrm{N}(\mathrm{QC}(S)^{\circ})$ [21]. Following [37] and [4], for an algebraic stack $\mathcal{X}$ we define the stable ∞-category $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ of quasi-coherent complexes by

$$\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) := \lim_{S \to \mathcal{X}} \mathcal{D}_{\mathrm{qcoh}}(S)$$

where lim means a limit in the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ of large ∞-categories, and the limit is taken over all affine schemes S over $\mathcal{X}$. For a morphism $f : S \to \mathcal{X}$, we define $f^* : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(S)$ to be the natural projection $\lim_{S \to \mathcal{X}} \mathcal{D}_{\mathrm{qcoh}}(S) \to \mathcal{D}_{\mathrm{qcoh}}(S)$. Since $\mathcal{D}_{\mathrm{qcoh}}(S)$ is a presentable ∞-category for any affine scheme S, thus by [18, 5.5.3.13] $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ is presentable.

Let $\mathcal{D}_{\mathrm{qcoh}}(R) := \mathcal{D}_{\mathrm{qcoh}}(\operatorname{Spec} R)$. Let $\mathcal{X}$ be an algebraic stack over R. Let J be the category of R-affine schemes over $\mathcal{X}$ and we abuse notation and often write J for the nerve $\mathrm{N}(J)$ of J. Let $\widehat{\mathrm{Cat}}_{\Delta}$ be the category of large simplicial categories. Let $J^{op} \to \widehat{\mathrm{Cat}}_{\Delta}$ be a functor which sends $S \to \mathcal{X} \in J$ to $\mathrm{QC}(S)^{\circ}$ and sends $f : S' \to S$ (over $\mathcal{X}$) to $f^* : \mathrm{QC}(S)^{\circ} \to \mathrm{QC}(S')^{\circ}$. Note that $\mathrm{QC}(S)^{\circ}$ are fibrant simplicial categories [18, 1.1.4.3]. By the simplicial nerve functor $\mathrm{N} : \widehat{\mathrm{Cat}}_{\Delta} \to \widehat{\mathrm{Set}}_{\Delta}$, we have $J^{op} \to \widehat{\mathrm{Cat}}_{\infty}$ (and $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ is a limit of the diagram). Let $\widehat{\mathrm{Set}}_{\Delta}^{+}$ denote the category of marked large simplicial sets [18, 3.1], which is endowed with Cartesian model structure. This model

category is equipped with monoidal structure given by Cartesian product $\mathcal{C} \times \mathcal{D}$ [22, 4.1], and it gives rise to a monoidal structure on $\mathrm{N}(\widehat{\mathrm{Set}}_\Delta^+) \simeq \widehat{\mathcal{C}\mathrm{at}}_\infty$. Let $\mathcal{M}\mathrm{od}_{\mathcal{D}_{\mathrm{qcoh}}(R)}$ be the ∞-category of left module objects in $\widehat{\mathcal{C}\mathrm{at}}_\infty$ over the monoidal ∞-category $\mathcal{D}_{\mathrm{qcoh}}(R)$ (see [22, Section 2, 4.1]). By the construction there is a natural functor

$$J^{op} \longrightarrow \mathcal{M}\mathrm{od}_{\mathcal{D}_{\mathrm{qcoh}}(R)}$$

which extends $J^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Let $\mathcal{P}r^L$ be the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ spanned by presentable ∞-categories, in which functors are left adjoints (see [18, 5.5.3]). The ∞-category $\mathcal{P}r^L$ inherits the (symmetric) monoidal structure described in [22, 4.1]. Let $\mathcal{M}\mathrm{od}_{\mathcal{D}_{\mathrm{qcoh}}(R)}(\mathcal{P}r^L)$ denote the ∞-category of left module objects in $\mathcal{P}r^L$ over the monoidal ∞-category $\mathcal{D}_{\mathrm{qcoh}}(R)$ (it is equivalent to the ∞-category $\mathrm{c}\mathcal{M}\mathrm{od}_{\mathcal{D}_{\mathrm{qcoh}}(k)}(\mathcal{P}r^L)$ of module objects of $\mathcal{P}r^L$ over $\mathcal{D}_{\mathrm{qcoh}}(R)$). Since the left module map $\mathcal{D}_{\mathrm{qcoh}}(R) \times \mathcal{D}_{\mathrm{qcoh}}(S) \to \mathcal{D}_{\mathrm{qcoh}}(S)$ is colimit-preserving separately in each variable, thus $J^{op} \to \mathcal{M}\mathrm{od}_{\mathcal{D}_{\mathrm{qcoh}}(R)}$ yields $J^{op} \to \mathcal{M}\mathrm{od}_{\mathcal{D}_{\mathrm{qcoh}}(R)}(\mathcal{P}r^L)$. Take a limit of $J^{op} \to \mathcal{M}\mathrm{od}_{\mathcal{D}_{\mathrm{qcoh}}(R)}(\mathcal{P}r^L)$. According to [22, 2.3.5] and [18, 5.5.3.13], the underlying category of this limit is equivalent to $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$. Thus $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ inherits an "R-linear structure" in $\mathcal{P}r^L$. (This structure is not needed until Section 5 except the application of derived Morita theory.)

By [23, 4.3], $\mathcal{D}_{\mathrm{qcoh}}(S)$ has a symmetric monoidal structure (arising from tensor products) for each affine scheme S. Applying [23, 4.17] to the symmetric monoidal ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ [23, 6.2], we see that $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ inherits a symmetric monoidal structure, where the tensor product is define pointwisely (cf. Section 5). The symmetric monoidal structure on $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ induces a symmetric monoidal structure on the homotopy category $\mathrm{D}_{\mathrm{qcoh}}(\mathcal{X}) = \mathrm{h}\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$.

2.3. **Schemes and stacks.** Let $k = R$ be a field. Let $\mathcal{X}$ be an algebraic stack over k. In the main results of this paper we will treat the following two cases:

(i) $\mathcal{X}$ is a noetherian scheme which has a very ample invertible sheaf (e.g., quasi-projective varieties).

(ii) $\mathcal{X}$ is a tame separated (Deligne-Mumford) algebraic stack of the form $[X/G]$ where X is a finitely generated noetherian scheme and G is a linear algebraic group acting on X. Suppose further that the coarse moduli space is quasi-projective and X has a G-ample invertible sheaf.

Remark 2.1. The following are examples of algebraic stacks which satisfy the condition (ii).

(1) GIT stable quotients whose stabilizer groups are all finite group. Let X be a separated scheme of finite type over a field, endowed with action of a linearly reductive group G. Assume that there exists a G-linearized ample invertible sheaf $\mathcal{L}$ and $X^s(\mathcal{L})$ is the open subset of stable points. Suppose furthermore that every stabilizer is finite. Then the quotient stack $[X^s(\mathcal{L})/G]$ satisfies the condition (ii). Such quotients often arise from Geometric Invariant Theory.

(2) Separated and smooth tame Deligne-Mumford stacks which satisfy the conditions: (a) it is generically a scheme, (b) the coarse moduli space is quasi-projective (see [39, Theorem 1.2]). (For example, toric stacks (orbifolds) cf. [11].)

(3) Moduli stacks of stable curves, stable maps, polarized abelian varieties, Calabi-Yau manifolds in characteristic zero.

Let us recall the notion of *perfect stacks* introduced in [4]. In loc. cit., the authors offer us the concept in the framework of derived stacks and prove derived Morita theory for perfect stacks, but here we consider only usual algebraic stacks. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{D}_{\mathrm{perf}}(\mathcal{X}) \subset \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ be the full subcategory consisting of perfect complexes. (A strictly perfect complex on $\mathcal{X}$ is a bounded complex of vector bundles. A complex is said to be perfect if locally on the smooth site of $\mathcal{X}$ it is quasi-isomorphic to a strictly perfect complex. According to [15, 3.6, 3.7] under the assumption of (i) and (ii) any perfect complex of $\mathcal{O}_{\mathcal{X}}$-modules is quasi-isomorphic to a perfect complex of quasi-coherent sheaves.) An algebraic stack $\mathcal{X}$ is said to be perfect if the ∞-category $\mathrm{Ind}\mathcal{D}_{\mathrm{perf}}(\mathcal{X})$ of Ind-objects [18, 5.3] of perfect complexes is naturally equivalent to $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$. A large class of stacks satisfies perfectness (e.g. quasi-compact and separated schemes, quotient stacks in characteristic zero, algebraic stacks satisfying (i) or (ii), etc, see [4], [35], Corollary 2.3). If $\mathcal{X}$ is a perfect stack, then $\mathcal{D}_{\mathrm{perf}}(\mathcal{X})$ is the full subcategory spanned by compact objects in $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ on one hand, and $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ is $\mathrm{Ind}\mathcal{D}_{\mathrm{perf}}(\mathcal{X})$ on the other hand. Consequently, we can transform $\mathcal{D}_{\mathrm{perf}}(\mathcal{X})$ into $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ and transform $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ into $\mathcal{D}_{\mathrm{perf}}(\mathcal{X})$ in the categorical fashion. In particular, we can consider an exact functor $\mathcal{D}_{\mathrm{perf}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{perf}}(S)$ to be a colimit-preserving functor $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(S)$, which preserves compact objects.

Lemma 2.2. *Let $\mathcal{X}$ be a tame Deligne-Mumford stack which is separated and of finite type over $\mathbb{Z}$. Suppose that its coarse moduli space is a scheme. Then compact and dualizable objects in $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ coincide (see Section 4 for the notion of dualizable objects).*

Proof. To see that dualizable objects are compact, it is enough to show that the (derived) global section functor $\Gamma(\mathcal{X}, -)$ preserves colimits since the functor $\mathrm{Hom}(P, -)$ is equivalent to $\Gamma(\mathcal{X}, P^* \otimes -)$ for any dualizable object P and the functor $P^* \otimes -$ preserves colimits. Here P^* is a dual of P. By our assumption on $\mathcal{X}$, we have a coarse moduli space $p : \mathcal{X} \to M$ such that M is quasi-compact and separated. Thus M is a perfect stack (cf. [4, Section 3]). Notice that the dualizable object $\mathcal{O}_M$ is compact in $\mathcal{D}_{\mathrm{qcoh}}(M)$. Since $\mathcal{O}_M$ is compact, the functor $\Gamma(M, -)$ preserves colimits. Hence to see that $\Gamma(\mathcal{X}, -)$ preserves colimits, it is sufficient to show that the pushforward p_* preserves colimits. There exist an étale surjective morphism $U \to M$ and a Cartesian diagram

$$
\begin{array}{ccc}
[W/G'] & \longrightarrow & \mathcal{X} \\
\downarrow{\scriptstyle p_U} & & \downarrow{\scriptstyle p} \\
U & \longrightarrow & M
\end{array}
$$

where U is an affine scheme and $[W/G']$ is a quotient stack of a finite scheme W (over U) by action of a finite (étale) group scheme G' over U. Then $[W/G']$ is perfect by [4, Proposition 3.26] (we here use the tameness of $\mathcal{X}$). Thus by [4, Proposition 3.10, 3.23] p_U is a perfect morphism and p_{U*} preserves small colimits. Since $U \to M$ is étale surjective, (using descent and base change theorem) we see that p_* also preserves small colimits. Conversely, to see that compact objects are dualizable, it is enough to repeat

the same argument in the proof of [4, Lemma 3.20] for $p : \mathcal{X} \to M$ and the affine covering map $U \to M$. $\square$

According to [4, Proposition 3.9] an algebraic stack $\mathcal{X}$ is perfect if and only if compact and dualizable objects in $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ coincide and $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ is compactly generated. The recent powerful result of the existence of compact generators by Toën show that a separated and quasi-compact Deligne-Mumford stack has a compact generator if its coarse moduli space is a scheme (see [35, 4.2]). Thus we have:

Corollary 2.3. *Let $\mathcal{X}$ be a tame Deligne-Mumford stack which is separated and of finite type over $\mathbb{Z}$. Suppose that a coarse moduli space for $\mathcal{X}$ is a scheme. Then $\mathcal{X}$ is a perfect stack.*

3. EXTENSION LEMMAS

In this section let $\mathcal{X}$ and S be perfect stacks. Let $\mathcal{D}_{\mathrm{vect}}(\mathcal{X})$ (resp. $\mathcal{D}_{\mathrm{vect}}(S)$) denote the full subcategory of $\mathcal{D}_{\mathrm{perf}}(\mathcal{X})$ (resp. $\mathcal{D}_{\mathrm{perf}}(S)$), spanned by quaisi-coherent complexes which are quasi-isomorphic to vector bundles placed in degree zero on $\mathcal{X}$ (resp. S). We will say that a algebraic stack $\mathcal{X}$ has cohomological dimension zero if $\mathrm{H}^i(\mathcal{X}, E)$ is zero for any quasi-coherent sheaf E and $i > 0$.

Lemma 3.1. *Suppose that S has cohomological dimension zero.*
  (i) *$\mathcal{D}_{\mathrm{vect}}(S)$ is equivalent to a 1-category (cf. [18, Section 2.3.4]).*
  (ii) *Let $\mathcal{E}$ be an ∞-category. The functor $\mathrm{Fun}(\mathrm{N}(\mathrm{h}\mathcal{E}), \mathcal{D}_{\mathrm{vect}}(S)) \to \mathrm{Fun}(\mathcal{E}, \mathcal{D}_{\mathrm{vect}}(S))$ associated to the projection $\mathcal{E} \to \mathrm{N}(\mathrm{h}\mathcal{E})$ is a categorical equivalence.*

Proof. We first prove (i). To prove this, notice that for any locally free sheaves E and F on S, the Ext-group $\mathrm{Ext}^i(E, F)$ is zero for $i \neq 0$. It follows that for every pair of objects $E, F \in \mathcal{D}_{\mathrm{vect}}(S)$, the mapping space $\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}(S)}(E, F)$ is discrete, that is, 0-truncated. Therefore $\mathcal{D}_{\mathrm{vect}}(S)$ is equivalent to a 1-category (cf. [18, 2.3.4.18]). The claim (ii) follows from [18, 2.3.4.12]. $\square$

The homotopy category $\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X})$ is a 1-category whose objects are vector bundles on $\mathcal{X}$, placed on degree zero. A morphism $E \to F$ in $\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X})$ can be considered to be a morphism of locally free sheaves on $\mathcal{X}$.

Lemma 3.2. *For any $n \geq 0$, let I_n denote the simplicial set defined as follows:*

$$
\begin{aligned}
I_0 &:= \Delta^0 \\
I_{n+1} &:= (I_n \times \Delta^1) \coprod_{I_n \times \Delta^{\{0\}}} (I_n \times \Delta^1).
\end{aligned}
$$

Let $P \in \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ be a strict perfect complex on $\mathcal{X}$ which lies in $(-\infty, 0]$. Suppose that P is represented by the complex of the form

$$
\dots 0 \to P^{-n} \to \dots \to P^{-1} \to P^0 \to 0 \to \dots
$$

where P^i is a vector bundle placed in degree i. Then there exists a set of diagrams $\{p_k : I_k \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})\}_{k \geq 0}$ such that the complex $(\sigma^{\leq -n+k}P)[-n+k]$ is a colimit of p_k for any $k \geq 0$ and the restriction $p_k|_{I_{k-1} \times \Delta^0}$ is p_{k-1} for any $k \geq 1$.

Proof. We will inductively construct p_k. Let p_0 be the map $\Delta^0 \to \mathcal{D}_{\text{qcoh}}(\mathcal{X})$ which sends $0 \in \Delta^0$ to P^{-n}. Now suppose that we have constructed p_k for any $k \leq l$. Let P' denote the complex $(\sigma^{\leq -n+l}P)[-n+l]$. Since P' is a colimit of p_l, the canonical morphism of complexes $P' \to P^{-n+l+1}$ induces a map $q : I_k \times \Delta^1 \to \mathcal{D}_{\text{qcoh}}(\mathcal{X})$ such that $q|_{I_k \times \Delta^{\{0\}}} = p_l$ and $q|_{I_k \times \Delta^{\{1\}}}$ is a constant diagram with value P^{-n+l+1}. Let $q' : I_k \times \Delta^1 \to \mathcal{D}_{\text{qcoh}}(\mathcal{X})$ be a map such that $q'|_{I_k \times \Delta^{\{0\}}} = p_l$ and $q'|_{I_k \times \Delta^{\{1\}}}$ is a constant diagram with value 0. Then we obtain a diagram $p_{l+1} : I_{l+1} \to \mathcal{D}_{\text{qcoh}}(\mathcal{X})$ by gluing q and q' along p_l. The (homotopy) pushout $0 \leftarrow P' \to P^{-n+l+1}$ is a colimit of p_{l+1} by [18, 4.4.2.2]. Consider the mapping cylinder

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P^{-n} & \xrightarrow{(-1)^{n-l}d_P} & \cdots & \longrightarrow & P^{-n+l-1} & \xrightarrow{(-1)^{n-l}d_P} & P^{-n+l} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \to & P^{-n} & \to & P^{-n} \oplus P^{-n+1} & \to & \cdots & \to & P^{-n+l-1} \oplus P^{-n+l} & \to & P^{-n+l} \oplus P^{-n+l+1} & \to & 0
\end{array}
$$

of $P' \to P^{-n+l+1}$. Here we regard P^{-n+l+1} as a complex whose degree zero term is P^{-n+l+1}. Let P'' denote the lower complex. The vertical arrows in the mapping cylinder are split monomorphisms and thus $(\sigma^{\leq -n+l+1}P)[-n+l+1]$ is a homotopy colimit of the diagram $0 \leftarrow P' \to P''$ (cf. [21, 13.4]). $\qquad\square$

Remark 3.3. An analogues result holds for a bounded complex of quasi-coherent sheaves $P^\bullet$ such that $P^i = 0$ for $i > 0$.

Lemma 3.4. *Let $\mathcal{X}$ be an algebraic stack which satisfies either the condition (i) or (ii) in Section 2.3. Let P be a complex of quasi-coherent sheaves, i.e., $P \in \mathcal{D}_{\text{qcoh}}(\mathcal{X})$. Then there exist a filtered system of complexes $\{E(n,m)\}_{n \geq 0, m \geq 0}$ and a quasi-isomorphism $\lim_{n,m} E(n,m) \to P$ such that $E(n,m)$ is quasi-isomorphic to $\sigma^{\geq -m}\tau^{\leq n}P$, $E(n,m)$ is a complex which in each degree is an infinite direct sum of invertible sheaves, and $E(n,m)^i$ is zero for $i > n$ and $i < -m$.*

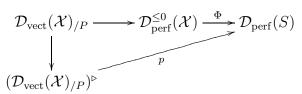*Proof.* It follows from [33, 2.3.2] and its proof. $\qquad\square$

Let $\mathcal{D}_{\text{perf}}^{\leq 0}(\mathcal{X})$ denote the full subcategory of $\mathcal{D}_{\text{perf}}(\mathcal{X})$ spanned by complexes which are equivalent to complexes $P^\bullet$ such that if $i \leq 0$, P^i is a vector bundle, and if $i > 0$ then $P^i = 0$. The theory of left Kan extensions [18, 4.3.2.16] for ∞-categories provides the following lemma:

Lemma 3.5. *Let $\kappa : \text{Fun}(\mathcal{D}_{\text{perf}}^{\leq 0}(\mathcal{X}), \mathcal{D}_{\text{qcoh}}(S)) \to \text{Fun}(\mathcal{D}_{\text{vect}}(\mathcal{X}), \mathcal{D}_{\text{qcoh}}(S))$ be the functor induced by $\mathcal{D}_{\text{vect}}(\mathcal{X}) \subset \mathcal{D}_{\text{perf}}^{\leq 0}(\mathcal{X})$. Let $\mathcal{K}' \subset \text{Fun}(\mathcal{D}_{\text{vect}}(\mathcal{X}), \mathcal{D}_{\text{qcoh}}(S))$ be the full subcategory spanned by functors $\mathcal{D}_{\text{vect}}(\mathcal{X}) \to \mathcal{D}_{\text{qcoh}}(S)$ whose essential images lie in $\mathcal{D}_{\text{vect}}(S)$. Let $\mathcal{K} \subset \text{Fun}(\mathcal{D}_{\text{perf}}^{\leq 0}(\mathcal{X}), \mathcal{D}_{\text{qcoh}}(S))$ be the full subcategory spanned by the functors $\Phi : \mathcal{D}_{\text{perf}}^{\leq 0}(\mathcal{X}) \to \mathcal{D}_{\text{qcoh}}(S)$ such that Φ is a left Kan extension of $\Phi|_{\mathcal{D}_{\text{vect}}(\mathcal{X})} : \mathcal{D}_{\text{vect}}(\mathcal{X}) \to \mathcal{D}_{\text{qcoh}}(S)$ and $\kappa(\Phi) \in \mathcal{K}'$. Then the induced map $\mathcal{K} \to \mathcal{K}'$ is a categorical equivalence.*

• In what follows we will assume that $\mathcal{X}$ has the *resolution property*, that is, every coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ admits a surjective morphism $\mathcal{E} \to \mathcal{F}$ from a vector bundle $\mathcal{E}$. Under the condition (i) and (ii) in Section 2.3 (the existence of a G-ample invertible sheaf), $\mathcal{X}$ has the resolution property. However, note that the resolution property is not needed in Lemma 3.7, 3.12.

Proposition 3.6. *Suppose that $\mathcal{X}$ has cohomological dimension zero. Let $\overline{\Phi}$ be a colimit-preserving functor $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(S)$ such that $\overline{\Phi}(\mathcal{D}_{\mathrm{perf}}(\mathcal{X}))$ lies in $\mathcal{D}_{\mathrm{perf}}(S)$. Let $\Phi := \overline{\Phi}|_{\mathcal{D}_{\mathrm{perf}}^{\leq 0}(\mathcal{X})} : \mathcal{D}_{\mathrm{perf}}^{\leq 0}(\mathcal{X}) \to \mathcal{D}_{\mathrm{perf}}(S)$ and suppose that $\Phi(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$ lies in $\mathcal{D}_{\mathrm{vect}}(S)$. Then Φ belongs to $\mathcal{K}$.*

Proof. It is clear that $\kappa(\Phi)$ belongs to $\mathcal{K}'$. Thus it suffices to prove that Φ is a left Kan extension of $\Phi_0 := \Phi|_{\mathcal{D}_{\mathrm{vect}}(\mathcal{X})} : \mathcal{D}_{\mathrm{vect}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{perf}}(S)$. Recall that Φ is said to be a left Kan extension if for any $P \in \mathcal{D}_{\mathrm{perf}}^{\leq 0}(\mathcal{X})$ the induced functor p in the commutative diagram

$$\mathcal{D}_{\mathrm{vect}}(\mathcal{X})_{/P} \longrightarrow \mathcal{D}_{\mathrm{perf}}^{\leq 0}(\mathcal{X}) \xrightarrow{\Phi} \mathcal{D}_{\mathrm{perf}}(S)$$
$$\downarrow \qquad\qquad \nearrow p$$
$$(\mathcal{D}_{\mathrm{vect}}(\mathcal{X})_{/P})^{\triangleright}$$

is a colimit diagram. (Here the cone point of $(\mathcal{D}_{\mathrm{vect}}(\mathcal{X})_{/P})^{\triangleright}$ maps to $\Phi(P)$.) To prove this, we may replace $\mathcal{D}_{\mathrm{perf}}(S)$ by $\mathcal{D}_{\mathrm{qcoh}}(S)$. Fix a perfect complex P in $\mathcal{D}_{\mathrm{perf}}^{\leq 0}(\mathcal{X})$. It is quasi-isomorphic to a strict perfect complex since we impose the resolution property. Since $\overline{\Phi}$ preserves small colimits, it is enough to show that P is a colimit of $\mathcal{D}_{\mathrm{vect}}(\mathcal{X})_{/P} \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$. To this end, let $K = \mathcal{D}_{\mathrm{vect}}(\mathcal{X})$ and take a colimit R of the diagram $K_{/P} \to K \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$. By Lemma 3.2, P is a colimit of the diagram $p_n : I_n \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ of vector bundles (we here use the notation in Lemma 3.2). Invoking the universality of P and R, we obtain morphisms $P \to R$ and $R \to P$. Note that the composite $P \to R \to P$ is equivalent to the identity morphism. Since $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ is idempotent complete, R has the form $P \oplus P'$, and P is identified with the direct summand $P \oplus \{0\} \subset R$. We may and will identify R with $P \oplus P'$. To complete the proof, it will suffice to prove that P' is a zero object. Now suppose that P' is not a zero object. Then there exists $(\theta : E \to P) \in K_{/P}$ such that the corresponding morphism $\xi : E \to R$, in the colimit diagram $(K_{/P})^{\triangleright} \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$, whose cone point maps to R, induces a non-null-homotopic morphism $E \to R \simeq P \oplus P' \xrightarrow{\mathrm{pr}_2} P'$. On the other hand, since cohomological dimension of $\mathcal{X}$ is zero, there exists some $E \to P^0$ which represents $\theta : E \to P$. Note that the composite $P^0 \to P \oplus \{0\} \subset P \oplus P' \xrightarrow{\mathrm{pr}_2} P'$ is null-homotopic. It follows that $\xi : E \to P \oplus P'$ factors through $E \to P^0 \to P$. Consequently, $E \xrightarrow{\xi} P \oplus P' \to P'$ is null-homotopic. It gives rise to a contradiction, as desired. $\square$

For the ease of notation, in the proofs, we usually denote by $\mathcal{C}$ and $\mathcal{D}$ stable presentable ∞-categories $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ and $\mathcal{D}_{\mathrm{qcoh}}(S)$ respectively. Similarly, we denote by $\mathcal{C}_{\circ} \subset \mathcal{C}$ $\mathcal{D}_{\circ} \subset \mathcal{D}$ the full subcategories consisting of perfect complexes. (Note that $\mathrm{Ind}\mathcal{C}_{\circ} \simeq \mathcal{C}$ and $\mathrm{Ind}\mathcal{D}_{\circ} \simeq \mathcal{D}$.) Let $\mathcal{C}_v$ and $\mathcal{D}_v$ be the full subcategories of $\mathcal{C}$ and $\mathcal{D}$ respectively, spanned by vector bundles (i.e., complexes which are equivalent to vector bundles).

Lemma 3.7. *Let $\mathrm{Fun}'(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{qcoh}}(S)) \subset \mathrm{Fun}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{qcoh}}(S))$ be the full subcategory spanned by functors which are colimit-preserving in each variable. (the product $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})^{\times n}$ is n-times (homotopy) product.) Let $\mathrm{Fun}''(\mathcal{D}_{\mathrm{perf}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{perf}}(S))$ be the full subcategory of $\mathrm{Fun}(\mathcal{D}_{\mathrm{perf}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{perf}}(S))$, spanned by functors which preserve*

finite colimits in each variable. Then the restriction functor

$$\mathrm{Fun}'(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{qcoh}}(S)) \to \mathrm{Fun}''(\mathcal{D}_{\mathrm{perf}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{qcoh}}(S))$$

is a categorical equivalence.

Let $\mathrm{Fun}^{\diamond}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{qcoh}}(S))$ *be the full subcategory of* $\mathrm{Fun}'(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{qcoh}}(S))$, *spanned by functors which are compatible with full subcategories* $\mathcal{D}_{\mathrm{perf}}(\mathcal{X})^{\times n}$ *and* $\mathcal{D}_{\mathrm{perf}}(S)$. *Then the restriction functor*

$$\mathrm{Fun}^{\diamond}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{qcoh}}(S)) \to \mathrm{Fun}''(\mathcal{D}_{\mathrm{perf}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{perf}}(S))$$

is a categorical equivalences.

Proof. We first consider the case of $n = 1$. According to [18, 5.3.5.10] we have an equivalence $\mathrm{Fun}_{\mathrm{cont}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}_{\circ}, \mathcal{D})$, where $\mathrm{Fun}_{\mathrm{cont}}(\mathcal{C}, \mathcal{D})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors that preserve filtered colimits. Using [18, 5.3.5.15] we see that if $\Phi \in \mathrm{Fun}_{\mathrm{cont}}(\mathcal{C}, \mathcal{D})$ is a left Kan extension of $\phi \in \mathrm{Fun}''(\mathcal{C}_{\circ}, \mathcal{D})$, then Φ preserves (co)kernels, that is, Φ is colimit-preserving. Since the inclusion $\mathcal{C}_{\circ} \to \mathcal{C}$ is exact, thus we have an equivalence $\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}''(\mathcal{C}_{\circ}, \mathcal{D})$.

Now suppose that our claim holds in the case $n = l$. We will show that the injectivity of $\mathrm{Fun}'(\mathcal{C}^{\times(l+1)}, \mathcal{D}) \to \mathrm{Fun}''(\mathcal{C}_{\circ}^{\times(l+1)}, \mathcal{D})$. Let $\mathcal{P}r^{L}$ be the ∞-category of presentable categories in which functors are left adjoints. Then by [22, Section 4] and [23], $\mathcal{P}r^{L}$ has a symmetric monoidal structure ($\otimes$ denotes the tensor operation). Then we have equivalences

$$\mathrm{Fun}'(\mathcal{C}^{\times(l+1)}, \mathcal{D}) \simeq \mathrm{Fun}^{L}(\mathcal{C}^{\otimes(l+1)}, \mathcal{D}) \simeq \mathrm{Fun}^{L}(\mathcal{C}, \mathrm{Fun}^{L}(\mathcal{C}^{\otimes l}, \mathcal{D})) \simeq \mathrm{Fun}^{L}(\mathcal{C}, \mathrm{Fun}''(\mathcal{C}_{\circ}^{\times l}, \mathcal{D})).$$

The notation $\mathrm{Fun}^{L}(\ ,\ )$ indicates the full subcategory spanned by left adjoints and $\mathcal{C}^{\otimes(l+1)}$ is the $(l+1)$-times product $\mathcal{C} \otimes \cdots \otimes \mathcal{C}$. The above first equivalence follows from the definition of $\mathcal{C} \otimes \mathcal{C}$ (see [22, 4.1]). The second equivalence follows from the closed monoidal structure of $\mathcal{P}r^{L}$ (cf. [18, 5.5.3.9]). The third one follows from the case of $n = l$. Then by [18, 5.5.3.10] again $\mathrm{Fun}_{\mathrm{cont}}(\mathcal{C}, \mathrm{Fun}(\mathcal{C}_{\circ}^{\times l}, \mathcal{D})) \simeq \mathrm{Fun}(\mathcal{C}_{\circ}, \mathrm{Fun}(\mathcal{C}_{\circ}^{\times l}, \mathcal{D}))$ and they contain $\mathrm{Fun}^{L}(\mathcal{C}, \mathrm{Fun}''(\mathcal{C}_{\circ}^{\times l}, \mathcal{D}))$ as full subcategories. Thus we have a fully faithful functor $\mathrm{Fun}'(\mathcal{C}^{\times(l+1)}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_{\circ}^{\times(l+1)}, \mathcal{D})$. Next we show that the surjectivity of $\mathrm{Fun}'(\mathcal{C}^{\times(l+1)}, \mathcal{D}) \to \mathrm{Fun}''(\mathcal{C}_{\circ}^{\times(l+1)}, \mathcal{D})$. Since $\mathcal{C}_{\circ} \to \mathcal{C}$ is exact, the essential image of $\mathrm{Fun}'(\mathcal{C}^{\times(l+1)}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_{\circ}^{\times(l+1)}, \mathcal{D})$ lies in $\mathrm{Fun}''(\mathcal{C}_{\circ}^{\times(l+1)}, \mathcal{D})$. Let $\mathcal{C}^{\times(l+1)} \to \mathcal{D}$ be a left Kan extension of $\mathcal{C}_{\circ}^{\times(l+1)} \to \mathcal{D}$, which preserves filtered colimits separately in each variable of $\mathcal{C} \times \cdots \times \mathcal{C}$. To prove that $\mathrm{Fun}'(\mathcal{C}^{\times(l+1)}, \mathcal{D}) \to \mathrm{Fun}''(\mathcal{C}_{\circ}^{\times(l+1)}, \mathcal{D})$ is essentially surjective, it is enough to observe that if $\mathcal{C}_{\circ}^{\times(l+1)} \to \mathcal{D}$ preserves finite colimits separately in each variable, then the Kan extension $\mathcal{C}^{\times(l+1)} \to \mathcal{D}$ preserves colimits separately in each variable. It suffices to check it in each variable separately. Thus it follows from the case of $n = 1$.

Now the latter assertion is clear because $\mathcal{C}_{\circ} \to \mathcal{C}$ and $\mathcal{D}_{\circ} \to \mathcal{D}$ are exact. $\square$

Lemma 3.8. *Let $\mathcal{E}$ be a stable ∞-category. The inclusion $\mathcal{D}_{\mathrm{perf}}^{\leq 0}(\mathcal{X}) \to \mathcal{D}_{\mathrm{perf}}(\mathcal{X})$ induces a fully faithful functor*

$$\mathrm{Fun}''(\mathcal{D}_{\mathrm{perf}}(\mathcal{X}), \mathcal{E}) \to \mathrm{Fun}(\mathcal{D}_{\mathrm{perf}}^{\leq 0}(\mathcal{X}), \mathcal{E}).$$

Proof. It follows along the same lines as the proof of [21, 16.9]. $\square$

Let $\mathrm{Map}_\dagger(\mathcal{D}_{\mathrm{perf}}(\mathcal{X}), \mathcal{D}_{\mathrm{perf}}(S))$ denote the full subcategory spanned by exact functors $\Phi : \mathcal{D}_{\mathrm{perf}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{perf}}(S)$ such that $\Phi(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$ lies in $\mathcal{D}_{\mathrm{vect}}(S)$. By Lemma 3.5 and 3.8 together with Proposition 3.6, we obtain:

Corollary 3.9. *Suppose that $\mathcal{X}$ has cohomological dimension zero. Then the functor*

$$\mathrm{Map}_\dagger(\mathcal{D}_{\mathrm{perf}}(\mathcal{X}), \mathcal{D}_{\mathrm{perf}}(S)) \to \mathrm{Map}(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}), \mathcal{D}_{\mathrm{vect}}(S))$$

is a fully faithful functor.

Lemma 3.10. *Let $\mathcal{E}$ be a stable ∞-category. Then the natural functor*

$$\mathrm{Fun}''(\mathcal{D}_{\mathrm{perf}}(\mathcal{X})^{\times n}, \mathcal{E}) \to \mathrm{Fun}((\mathcal{D}_{\mathrm{perf}}(\mathcal{X})^{\leq 0})^{\times n}, \mathcal{E})$$

is fully faithful.

Proof. Let $\mathcal{C}_\circ^{\leq 0} = \mathcal{D}_{\mathrm{perf}}^{\leq 0}(\mathcal{X})$. The case of $n = 1$ follows from Lemma 3.8. Now suppose that the case of $n = l$ holds. There are natural fully faithful functors

$$\mathrm{Fun}''(\mathcal{C}_\circ^{\times(l+1)}, \mathcal{E}) \to \mathrm{Fun}''(\mathcal{C}_\circ, \mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})) \to \mathrm{Fun}(\mathcal{C}_\circ^{\leq 0}, \mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})).$$

The second functor is fully faithful by Lemma 3.8 and the fact that $\mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})$ is stable. The essential image of $\mathrm{Fun}''(\mathcal{C}_\circ^{\times(l+1)}, \mathcal{E})$ lies in $\mathrm{Fun}(\mathcal{C}_\circ^{\leq 0}, \mathrm{Fun}''(\mathcal{C}_\circ^{\times l}, \mathcal{E}))$. In addition by the case of $n = l$ we have a fully faithful functor

$$\mathrm{Fun}(\mathcal{C}_\circ^{\leq 0}, \mathrm{Fun}''(\mathcal{C}_\circ^{\times l}, \mathcal{E})) \to \mathrm{Fun}(\mathcal{C}_\circ^{\leq 0}, \mathrm{Fun}((\mathcal{C}_\circ^{\leq 0})^{\times l}, \mathcal{E})).$$

Since $\mathrm{Fun}(\mathcal{C}_\circ^{\leq 0}, \mathrm{Fun}((\mathcal{C}_\circ^{\leq 0})^{\times l}, \mathcal{D}_\circ)) \simeq \mathrm{Fun}((\mathcal{C}_\circ^{\leq 0})^{\times(l+1)}, \mathcal{D}_\circ)$, thus the case of $n = l + 1$ follows. $\square$

Lemma 3.11. *Suppose that $\mathcal{X}$ has cohomological dimension zero. Then the restriction*

$$\mathrm{Fun}''(\mathcal{D}_{\mathrm{perf}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{perf}}(S)) \to \mathrm{Fun}(\mathcal{D}_{\mathrm{vect}}(\mathcal{X})^{\times n}, \mathcal{D}_{\mathrm{perf}}(S))$$

is a fully faithful functor.

Proof. Let $\mathcal{E}$ be a stable presentable ∞-category. We may replace $\mathcal{D}_{\mathrm{perf}}(S)$ by $\mathcal{E}$ (consider $\mathrm{Ind}\mathcal{D}_{\mathrm{perf}}(S)$). We first consider the case of $n = 1$. By Lemma 3.6 (and its proof), for any $P \in \mathcal{C}_\circ^{\leq 0}$, P is a colimit of the natural diagram $(\mathcal{C}_v)_{/P} \to \mathcal{C}$. Since any object F in the full subcategory $\mathrm{Fun}''(\mathcal{C}_\circ, \mathcal{E}) \subset \mathrm{Fun}(\mathcal{C}_\circ^{\leq 0}, \mathcal{E})$ (cf. Lemma 3.10) extends to a colimit-preserving functor $\mathcal{C} \to \mathcal{E}$ by Lemma 3.7, F is a left Kan extension of $F|_{\mathcal{C}_v}$. Thus we have a fully faithful embedding $\mathrm{Fun}''(\mathcal{C}_\circ, \mathcal{E}) \subset \mathrm{Fun}(\mathcal{C}_v, \mathcal{E})$ induced by the inclusion $\mathcal{C}_v \to \mathcal{C}_\circ$.

Next suppose that the case of $n = l$ holds. We have fully faithful functors

$$\mathrm{Fun}''(\mathcal{C}_\circ^{\times(l+1)}, \mathcal{E}) \to \mathrm{Fun}''(\mathcal{C}_\circ, \mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})) \to \mathrm{Fun}(\mathcal{C}_\circ^{\leq 0}, \mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})).$$

By the observation in the case of $n = 1$ (note that $\mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})$ is stable and presentable), we have a fully faithful embedding

$$\mathrm{Fun}''(\mathcal{C}_\circ, \mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})) \subset \mathrm{Fun}(\mathcal{C}_v, \mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})).$$

Note that if a functor $F : \mathcal{C}_v \to \mathrm{Fun}(\mathcal{C}_\circ^{\times l}, \mathcal{E})$ belongs to the essential image of $\mathrm{Fun}''(\mathcal{C}_\circ^{\times(l+1)}, \mathcal{E})$ then $F(\mathcal{C}_v)$ maps to $\mathrm{Fun}''(\mathcal{C}_\circ^{\times l}, \mathcal{E})$. Using the case of $n = l$ we also have a fully faithful embedding $\mathrm{Fun}(\mathcal{C}_v, \mathrm{Fun}''(\mathcal{C}_\circ^{\times l}, \mathcal{E})) \subset \mathrm{Fun}(\mathcal{C}_v^{\times(l+1)}, \mathcal{E})$. This completes the proof. $\square$

Lemma 3.12. *Let $j : \mathcal{U} \to \mathcal{X}$ be a quasi-compact open immersion. Then the restriction functor $j^* : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{U})$ and $j_* : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{U}) \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ induce a localization (cf. [18, 5.2.7.2])*

$$j^* : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \rightleftarrows \mathcal{D}_{\mathrm{qcoh}}(\mathcal{U}) : j_*.$$

Proof. Let T be a collection of morphisms $F \to F' \in \mathrm{Fun}(\Delta^1, \mathcal{C})$ which are quasi-isomorphisms on $\mathcal{U}$. We let $T^{-1}\mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by T-local objects, that is, objects $F \in \mathcal{C}$ such that $\mathrm{Map}_{\mathcal{C}}(E', F) \to \mathrm{Map}_{\mathcal{C}}(E, F)$ is a homotopy equivalence for any $E \to E' \in T$. We claim that $\mathcal{C}'$ is equivalent to $T^{-1}\mathcal{C}$. More precisely, we will observe that $j_* : \mathcal{C}' \to \mathcal{C}$ factors through $T^{-1}\mathcal{C}$ and it is a homotopy inverse of $j^* : T^{-1}\mathcal{C} \to \mathcal{C}'$. To see that j_*E is a T-local object for any $E \in \mathcal{C}'$, it suffices to show that, if $F' \to F$ in $\mathcal{C}$ is a quasi-isomorphism on $\mathcal{U}$, then the induced functor $\mathrm{Map}_{\mathcal{C}}(F, j_*E) \to \mathrm{Map}_{\mathcal{C}}(F', j_*E)$ is a weak homotopy equivalence. This equivalence follows from weak homotopy equivalences

$$\mathrm{Map}_{\mathcal{C}}(F, j_*E) \simeq \mathrm{Map}_{\mathcal{C}'}(j^*F, E), \ \mathrm{Map}_{\mathcal{C}}(F', j_*E) \simeq \mathrm{Map}_{\mathcal{C}'}(j^*F', E)$$

induced by the adjunction, and $\mathrm{Map}_{\mathcal{C}'}(j^*F, E) \simeq \mathrm{Map}_{\mathcal{C}'}(j^*F', E)$. Hence j_*E is a T-local object. It remains to show that $j_* : \mathcal{C}' \to T^{-1}\mathcal{C}$ is a homotopy inverse of $j^* : T^{-1}\mathcal{C} \to \mathcal{C}'$. For any $E \in \mathcal{C}'$ the adjoint map $j^*j_*E \to E$ is a quasi-isomorphism (j is an open immersion). For any $F \in T^{-1}\mathcal{C}$, the adjoint map $F \to j_*j^*F$ is a quasi-isomorphism on $\mathcal{U}$ (this means that $F \to j_*j^*F$ is an equivalence in $T^{-1}\mathcal{C}$). Thus $T^{-1}\mathcal{C} \simeq \mathcal{C}'$.

$\square$

4. Symmetric monoidal functors and Derived Morita theory

Let $\mathcal{X}$ be an algebraic stack over a field k and let S be a scheme over k. Let $\Phi : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(S)$ be a k-linear symmetric monoidal functor which preserves small colimits. We first give a condition under which Φ preserves vector bundles, i.e., $\Phi(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$ lies in $\mathcal{D}_{\mathrm{vect}}(S)$.

Let us recall the notions of *integral functors* and their *integral kernels*. An object $P \in \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X} \times S)$ gives rise to an exact functor $\Phi_P := \mathrm{pr}_{2*}(\mathrm{pr}_1^*(-) \otimes P)$ where pr_1 and pr_2 denote the natural projections from $\mathcal{X} \times S$ to $\mathcal{X}$ and S respectively. The functor Φ_P is called *integral functor* and P is called an *integral kernel*, or simply *kernel* of Φ_P. To avoid unnecessary confusion we often denote by $\otimes^{\mathbf{L}}$ the derived tensor operation and denote by $\otimes$ the ordinary tensor operation. Similarly, $\mathbb{R}(\bullet)_*$ means the derived pushforward, whereas $(\bullet)_*$ indicates the ordinary pushforward. Moreover, to emphasize that an object is a cochain complex we often write $P^\bullet, Q^\bullet, \ldots$ for cochain complexes. We write $\mathrm{D}_{\mathrm{qcoh}}(\bullet)$ for the homotopy category $\mathrm{h}\mathcal{D}_{\mathrm{qcoh}}(\bullet)$.

Proposition 4.1. *Let $\mathcal{X}$ be an algebraic stack over k. Suppose that the cohomological dimension of $\mathcal{X}$ is finite, i.e., there exists an integer d such that for any quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module F and $q > d$, we have $\mathrm{H}^q(\mathcal{X}, F) = 0$. Let S be a scheme over k. Let $\Phi : \mathrm{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathrm{D}_{\mathrm{qcoh}}(S)$ be a symmetric monoidal functor whose underlying functor is an integral functor induced by a bounded kernel $P \in \mathrm{D}_{\mathrm{qcoh}}^{\mathrm{b}}(\mathcal{X} \times_k S)$. Then Φ preserves vector bundles.*

Before the proof of this proposition, let us recall the notion of *dualizable* objects in a symmetric monoidal category. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a (ordinary) symmetric monoidal

category. An object M in $\mathcal{C}$ is called *dualizable* if there exist an object $M^* \in \mathcal{C}$ and morphisms $\eta : \mathbb{1} \to M \otimes M^*$ and $\epsilon : M^* \otimes M \to \mathbb{1}$ satisfying the following conditions:

- The composite $M \xrightarrow{\eta \otimes \mathrm{id}_M} M \otimes M^* \otimes M \xrightarrow{\mathrm{id}_M \otimes \epsilon} M$ is the identity map,
- The composite $M^* \xrightarrow{\mathrm{id}_{M^*} \otimes \eta} M^* \otimes M \otimes M^* \xrightarrow{\epsilon \otimes \mathrm{id}_{M^*}} M^*$ is the identity map.

The object M^* is called a *dual* of M. If M^* exists, it is unique up to isomorphism. If $\Psi : \mathcal{C} \to \mathcal{C}'$ is a symmetric monoidal functor and M is a dualizable object of $\mathcal{C}$, $\Psi(M)$ is also dualizable and $\Psi(M)^* \simeq \Psi(M^*)$. In the case that $\mathcal{C}$ is a category of quasi-coherent complexes, any perfect complex E is dualizable and its dual is isomorphic to the (usual) derived dual $\mathbb{R}\mathcal{H}om(E, \mathcal{O})$. Therefore, for any symmetric monoidal functor $\Phi : \mathrm{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathrm{D}_{\mathrm{qcoh}}(S)$ and any perfect complex $E \in \mathrm{D}_{\mathrm{qcoh}}(\mathcal{X})$, $\Phi(\mathbb{R}\mathcal{H}om(E, \mathcal{O}_{\mathcal{X}}))$ is isomorphic to $\mathbb{R}\mathcal{H}om(\Phi(E), \mathcal{O}_S)$.

Remark 4.2. Let $\mathcal{X}$ and S be algebraic stacks. Then any symmetric monoidal functor $F : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(S)$ preserves dualizable objects. According to [4, Proposition 3.6] dualizable objects and perfect complexes coincide in $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$. Also, dualizable objects and perfect complexes coincide in $\mathcal{D}_{\mathrm{qcoh}}(S)$. Consequently, any symmetric monoidal functor F preserves perfect complexes.

Proof of Proposition 4.1. We may and will assume that S is affine. Let d be the cohomological dimension of $\mathcal{X}$ and $m := \max\{ p \mid \mathrm{H}^p(P) \neq 0 \}$. To prove this proposition, we first claim that $\mathrm{H}^q(\Phi(E)) = 0$ for any vector bundles E on $\mathcal{X}$ and $q > m + d$. The category of quasi-coherent $\mathcal{O}_{\mathcal{X} \times S}$-modules has enough injective objects. (For the readers' convenience, we give an outline of the proof here. Let F be a quasi-coherent $\mathcal{O}_{\mathcal{X} \times S}$-module and let $p : U \to \mathcal{X} \times S$ be a smooth surjective map where U is an affine scheme. Take an injective quasi-coherent $\mathcal{O}_U$-module I which contains p^*F. Since p_*I is an injective $\mathcal{O}_{\mathcal{X} \times S}$-module, it is sufficient to check that the natural maps $F \to p_*p^*F$ and $p_*p^*F \to p_*I$ are injective. The first follows from the fact that p is faithfully flat and affine. The second is clear.) Hence there exists a bounded below complex of injective quasi-coherent $\mathcal{O}_{\mathcal{X} \times S}$-modules $I^\bullet$ which is quasi-isomorphic to P. Since $\mathrm{pr}_1^* E$ is a vector bundle, $\mathrm{pr}_1^* E \otimes I^\bullet$ is quasi-isomorphic to $\mathrm{pr}_1^* E \otimes^{\mathbf{L}} P$ and $\mathrm{pr}_1^* E \otimes I^l$ is an injective quasi-coherent $\mathcal{O}_{\mathcal{X} \times S}$-module for any $l \in \mathbb{Z}$. Thus, we have

$$(1) \qquad \mathrm{H}^q(\Phi(E)) = \mathrm{H}^q(\mathbb{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^* E \otimes^{\mathbf{L}} P)) \simeq \mathrm{H}^q(\mathrm{pr}_{2*}(\mathrm{pr}_1^* E \otimes I^\bullet)).$$

On the other hand, since $\mathrm{H}^l(I^\bullet) \simeq \mathrm{H}^l(P) = 0$ for any $l > m$ and $\mathrm{pr}_1^* E$ is flat, we have $\mathrm{H}^l(\mathrm{pr}_1^* E \otimes I^\bullet) \simeq \mathrm{pr}_1^* E \otimes \mathrm{H}^l(I^\bullet) = 0$ for any $l > m$. Hence the complex

$$(2) \qquad 0 \to \mathrm{pr}_1^* E \otimes Z^m \to \mathrm{pr}_1^* E \otimes I^m \to \mathrm{pr}_1^* E \otimes I^{m+1} \to \mathrm{pr}_1^* E \otimes I^{m+2} \to \cdots$$

is exact, where Z^m is $\ker(I^m \to I^{m+1})$. Moreover, since $\mathrm{pr}_1^* E \otimes I^l$ is injective for any $l \in \mathbb{Z}$, (2) gives an injective resolution of $\mathrm{pr}_1^* E \otimes Z^m$. Thus we have

$$(3) \qquad \mathrm{H}^q(\mathrm{pr}_{2*}(\mathrm{pr}_1^* E \otimes I^\bullet)) \simeq \mathrm{H}^{q-m}(\mathbb{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^* E \otimes Z^m)).$$

Since $q - m > d$, we have $\mathrm{H}^q(\mathrm{pr}_{2*}(\mathrm{pr}_1^* E \otimes I^\bullet)) \simeq \mathrm{H}^{q-m}(\mathbb{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^* E \otimes Z^m)) = 0$. Therefore we obtain $\mathrm{H}^q(\Phi(E)) = 0$ by (1) and (3).

We then show that $\mathrm{H}^q(\Phi(E)) = 0$ for any vector bundle E on $\mathcal{X}$ and $q > 0$. If $\Phi(E) = 0$, we have nothing to prove, so we assume that $\Phi(E) \neq 0$. Let l be the integer $\max\{ q \mid \mathrm{H}^q(\Phi(E)) \neq 0 \}$. In general, if F and G are objects in $\mathrm{D}_{\mathrm{perf}}(S)$ such that

$H^i(F) \simeq H^j(G) = 0$ for any $i > s$ and $j > t$, then we have $H^k(F \otimes^{\mathbf{L}} G) = 0$ for any $k > s + t$, and

$$(4) \qquad\qquad H^{s+t}(F \otimes^{\mathbf{L}} G) \simeq H^s(F) \otimes H^t(G).$$

(To see this, take a complex A (resp. B) which is quasi-isomorphic to F (resp. G) such that A^i is a *flat* $\mathcal{O}_S$-module for any $i \in \mathbb{Z}$ and $A^i = 0$ for any $i > s$ (resp. $B^j = 0$ for any $j > t$) and compute the cohomologies of the total complex of the double complex $A \otimes B$, which is quasi-isomorphic to $F \otimes^{\mathbf{L}} G$.) Hence for any positive integer n, we have $H^{nl}(\Phi(E)^{\otimes^{\mathbf{L}} n}) \simeq H^l(\Phi(E))^{\otimes n}$ ($\otimes n$ represents the n-times product). On the other hand, since Φ is symmetric monoidal, we have $\Phi(E)^{\otimes^{\mathbf{L}} n} \simeq \Phi(E^{\otimes^{\mathbf{L}} n})$. Hence we have $H^{nl}(\Phi(E^{\otimes^{\mathbf{L}} n})) \simeq H^l(\Phi(E))^{\otimes n}$. Since $H^l(\Phi(E))$ is a non-zero quasi-coherent sheaf of finite type by [33, 2.2.3], it follows that $H^l(\Phi(E))^{\otimes n} \neq 0$. Indeed, if $H^l(\Phi(E))^{\otimes n} = 0$, then $(H^l(\Phi(E)) \otimes k(s))^{\otimes n} \simeq H^l(\Phi(E))^{\otimes n} \otimes k(s) = 0$ for any point $s \in S$ where $k(s)$ denotes the residue field of s. This implies that $H^l(\Phi(E)) \otimes k(s) = 0$ since $H^l(\Phi(E)) \otimes k(s)$ is a $k(s)$-vector space. Hence the stalk $H^l(\Phi(E))_s$ is zero by Nakayama's lemma and so $H^l(\Phi(E)) = 0$. Therefore $H^{nl}(\Phi(E^{\otimes^{\mathbf{L}} n})) \simeq H^l(\Phi(E))^{\otimes n} \neq 0$. We have to show that l is not positive. If l is positive, there exists a positive integer n such that $nl > m + d$. In addition, since $E^{\otimes^{\mathbf{L}} n}$ is a locally free sheaf, $H^q(\Phi(E^{\otimes^{\mathbf{L}} n})) = 0$ for any $q > m + d$. It gives rise to a contradiction.

Next, we show that $H^{-q}(\Phi(E)) = 0$ for any $q > 0$. We have $\Phi(E) \simeq \Phi(E^{**}) \simeq \Phi(E^*)^* \simeq \mathbb{R}\mathcal{H}om(\Phi(E^*), \mathcal{O}_S)$. The second equivalence follows from the fact that Φ is symmetric monoidal. On the other hand, since E^* is a locally free sheaf, we have $H^q(\Phi(E^*)) = 0$ for any $q > 0$. Hence $\Phi(E^*)$ is quasi-isomorphic to a complex M such that $M^q = 0$ for any $q > 0$ and M^q is a free module for any q since S is affine. Thus we have

$$(5) \qquad\qquad H^{-q}(\mathbb{R}\mathcal{H}om(\Phi(E^*), \mathcal{O}_S)) \simeq H^{-q}(\mathcal{H}om(M, \mathcal{O}_S)) = 0.$$

Therefore we have $H^{-q}(\Phi(E)) = 0$ for any $q > 0$ by (5).

It remains to prove that $\Phi(E) \simeq H^0(\Phi(E))$ is a vector bundle. Since $H^0(\Phi(E))$ is quasi-coherent of finite type, it is enough to show that $H^0(\Phi(E))$ is flat. To see this, it is enough to show that $\mathcal{T}or_1^{\mathcal{O}_S}(H^0(\Phi(E)), N) = 0$ for any quasi-coherent $\mathcal{O}_S$-module N. We have

$$
\begin{aligned}
\mathcal{T}or_1^{\mathcal{O}_S}(H^0(\Phi(E)), N) \quad &\simeq \quad H^{-1}(H^0(\Phi(E)) \otimes^{\mathbf{L}} N) \\
&\simeq \quad H^{-1}(\Phi(E) \otimes^{\mathbf{L}} N) \\
&\simeq \quad H^{-1}(\mathbb{R}\mathcal{H}om(\Phi(E^*), N)) \\
&\simeq \quad H^{-1}(\mathbb{R}\mathcal{H}om(H^0(\Phi(E^*)), N)) = 0.
\end{aligned}
$$

Therefore $H^0(\Phi(E))$ is flat and it is a locally free sheaf. $\qquad\square$

Remark 4.3. We will apply Proposition 4.1 only to schemes $\mathcal{X}$ in this paper.

Remark 4.4. By the argument in the proof of Proposition 4.1, we see the following: if $\Phi(\mathcal{D}_{\text{vect}}(\mathcal{X}))$ is uniformly bounded above (i.e., there exists an integer a such that for any vector bundle E, $H^l(\Phi(E))$ is zero for any $l > a$), then $\Phi(E)$ is a vector bundle.

Let us recall one of key ingredients: *derived Morita theory* due to Toën, which was further generalized by Ben-Zvi, Francis and Nadler (see [34, Theorem 8.9], [4, Corollary

4.10]). We here recall the form which we can apply to our situation. Let $\mathcal{X}$ be a perfect stack over k. Then there is a natural functor

$$\text{(6)} \qquad \mathcal{D}_{\text{qcoh}}(\mathcal{X} \times_k S) \to \text{Fun}_{\mathcal{M}od_{\mathcal{D}_{\text{qcoh}}(k)}(\mathcal{P}r^L)}(\mathcal{D}_{\text{qcoh}}(\mathcal{X}), \mathcal{D}_{\text{qcoh}}(S)); \quad P \mapsto \Phi_P,$$

where $\mathcal{M}od_{\mathcal{D}_{\text{qcoh}}(k)}(\mathcal{P}r^L)$ is the ∞-category of left $\mathcal{D}_{\text{qcoh}}(k)$-modules in $\mathcal{P}r^L$ (see Section 2.2). Here $\mathcal{X} \times_k S$ is the fiber product in the category of ordinary stacks, but it coincides with the fiber product of derived stacks since k is a field.

Theorem 4.5 ([34], [4]). *Let $\mathcal{X}$ be a perfect algebraic stack over k. Then* (6) *gives a categorical equivalence.*

Proposition 4.6. *Let X be a noetherian scheme endowed with a very ample invertible sheaf over k and let S be a scheme over k. Let $\Phi : \mathrm{D}_{\text{qcoh}}(X) \to \mathrm{D}_{\text{qcoh}}(S)$ be a symmetric monoidal functor whose underlying functor is an integral functor induced by an integral kernel $P \in \mathrm{D}_{\text{qcoh}}(X \times_k S)$. Then P is a sheaf, that is, $\mathrm{H}^l(P) = 0$ for any $l \neq 0$.*

In the proof of this proposition, we consider derived pushforwards of *unbounded* complexes, so let us recall the notion of *K-injective* complexes (cf. [31]). A (unbounded) complex A in an abelian category $\mathcal{A}$ is called *K-injective* if, for any acyclic complex B in $\mathcal{A}$, the complex $\text{Hom}^\bullet_{\mathcal{A}}(B, A)$ is acyclic. If $\mathcal{A}$ is the category of quasi-coherent sheaves on a scheme, any complex in $\mathcal{A}$ is quasi-isomorphic to a K-injective complex. For any morphism f of schemes, the derived pushforward $\mathbb{R}f_*E$ of a complex E of quasi-coherent modules is quasi-isomorphic to the non-derived pushforward f_*I of a K-injective complex I which is quasi-isomorphic to E.

Proof of Proposition 4.6. This problem is local on S, we may assume that S is a connected affine scheme. Taking a K-injective resolution, we may assume that P is K-injective. For any $l \in \mathbb{Z}$, let d^l be the differential map $P^l \to P^{l+1}$ and $\alpha^l : \ker d^l \to \mathrm{H}^l(P)$ be the natural surjection. To prove this proposition, it is enough to show that $\alpha^l = 0$ for any integer $l \neq 0$ (since α^l is surjective).

Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X and let $Q(m)$ denote $Q \otimes \text{pr}_1^* \mathcal{O}_X(m)$ for any (unbounded) complex Q of quasi-coherent $\mathcal{O}_{X \times S}$-modules on $X \times S$. Fix $l \neq 0$. Now suppose that $\alpha^l \neq 0$. Then there exist $f \in \Gamma(X \times S, \mathcal{O}_{X \times S}(1))$ and $\phi \in \Gamma((X \times S)_f, \ker d^l)$ such that $\alpha^l|_{(X \times S)_f}(\phi) \neq 0$, where $(X \times S)_f$ denotes the affine open subscheme of $X \times S$ where f does not vanish. For any sufficiently large $n \in \mathbb{Z}$, $f^n \phi$ lies in $\Gamma(X \times S, \ker d^l(n))$ and thus it follows that $\Gamma(X \times S, \alpha^l(n)) \neq 0$, where $\alpha^l(n)$ denotes $\alpha^l \otimes \text{id}_{\mathcal{O}_{X \times S}(n)} : \ker d^l(n) \to \mathrm{H}^l(P)(n)$. Hence, to see that $\alpha^l = 0$, it is enough to show that the induced morphism $\Gamma(X \times S, \alpha^l(N)) : \Gamma(X \times S, \ker d^l(N)) \to \Gamma(X \times S, \mathrm{H}^l(P(N)))$ is zero for any sufficiently large $N \in \mathbb{Z}$. Since S is affine, this is equivalent to showing that the induced morphism $\text{pr}_{2*}(\alpha^l(N)) : \text{pr}_{2*}(\ker d^l(N)) \to \text{pr}_{2*}\mathrm{H}^l(P(N))$ is zero where pr_{2*} denotes the *non*-derived pushforward. Applying pr_{2*} to the complex

$$P(N) : \cdots \to P^{l-1}(N) \xrightarrow{d^{l-1}(N)} P^l(N) \xrightarrow{d^l(N)} P^{l+1}(N) \to \cdots$$

we obtain a complex

$$\text{pr}_{2*}(P(N)) : \cdots \to \text{pr}_{2*}P^{l-1}(N) \xrightarrow{\text{pr}_{2*}(d^{l-1}(N))} \text{pr}_{2*}P^l(N) \xrightarrow{\text{pr}_{2*}(d^l(N))} \text{pr}_{2*}P^{l+1}(N) \to \cdots .$$

From these complexes we have the following commutative diagram:

$$
\begin{array}{ccc}
\ker(\mathrm{pr}_{2*}d^l(N)) & \longrightarrow & \mathrm{H}^l(\mathrm{pr}_{2*}(P(N))) \\
\uparrow\simeq & & \downarrow \\
\mathrm{pr}_{2*}(\ker d^l(N)) & \xrightarrow{\ \mathrm{pr}_{2*}(\alpha^l(N))\ } & \mathrm{pr}_{2*}\mathrm{H}^l(P(N)).
\end{array}
$$

Hence, to show that $\mathrm{pr}_{2*}(\alpha^l(N)) = 0$, it is enough to show that $\mathrm{H}^l(\mathrm{pr}_{2*}(P(N))) = 0$ for any sufficiently large integer N. Since P is a K-injective complex and $\mathrm{pr}_1^*\mathcal{O}_X(N)$ is invertible, $P(N)$ is also a K-injective complex and hence $\mathrm{pr}_{2*}(P(N))$ is quasi-isomorphic to $\mathbb{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathcal{O}_X(N) \otimes^{\mathbf{L}} P)$. Moreover, since P is an integral kernel of Φ, we have $\Phi(\mathcal{O}_X(N)) \simeq \mathbb{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathcal{O}_X(N) \otimes^{\mathbf{L}} P)$. Hence $\Phi(\mathcal{O}_X(N))$ is quasi-isomorphic to the complex $\mathrm{pr}_{2*}(P(N))$. Thus, to show that $\mathrm{H}^l(\mathrm{pr}_{2*}(P(N))) = 0$, it will suffice to show that $\mathrm{H}^l(\Phi(\mathcal{O}_X(N))) = 0$. By [40, Theorem 2.3] and the connectedness of S, for any two objects F_1, $F_2 \in \mathrm{D}_{\mathrm{perf}}(S)$ such that $F_1 \otimes^{\mathbf{L}} F_2 \simeq \mathcal{O}_S$, there exist an invertible sheaf L on S and $m \in \mathbb{Z}$ such that $F_1 \simeq L[m]$. Since Φ preserves $\otimes^{\mathbf{L}}$ and structure sheaves, there exist an invertible sheaf L on S and $m \in \mathbb{Z}$ such that $\Phi(\mathcal{O}_X(1)) \simeq L[m]$ and we have $\Phi(\mathcal{O}_X(N)) \simeq L^{\otimes^{\mathbf{L}} N}[Nm]$. Since $\mathrm{H}^l(L^{\otimes^{\mathbf{L}} N}[Nm]) = 0$ if $l \neq -Nm$, thus $\mathrm{H}^l(\Phi(\mathcal{O}_X(N))) = 0$ for a sufficiently large $N \in \mathbb{Z}$. $\qquad\square$

Corollary 4.7. *Let $\mathcal{X}$ be a scheme that satisfies (i) in Section 2.3. Let S be a scheme over k. Let $\Phi : \mathrm{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}) \to \mathrm{D}_{\mathrm{qcoh}}^{\otimes}(S)$ be a symmetric monoidal whose underlying functor is an integral functor induced by an integral kernel in $\mathrm{D}_{\mathrm{qcoh}}(\mathcal{X} \times_k S)$. Then Φ preserves vector bundles.*

Next we consider the case (ii).

Proposition 4.8. *Let $\mathcal{X}$ be an algebraic stack that satisfies (ii) in Section 2.3. Let S be a scheme over k. Let $\Phi : \mathrm{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}) \to \mathrm{D}_{\mathrm{qcoh}}^{\otimes}(S)$ be a symmetric monoidal functor. Suppose that the underlying functor of Φ is an integral functor induced by an integral kernel in $\mathrm{D}_{\mathrm{qcoh}}(\mathcal{X} \times_k S)$. We abusively denote the integral functor by the same symbol $\Phi : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(S)$. Then Φ preserves vector bundles.*

Proof. For simplicity of notation, in this proof we denote by $\otimes$ (resp. f^*) the derived tensor operation (resp. derived pullback functor). We may suppose that S is affine.

Case 1. First we assume that k is algebraically closed and S is $\mathrm{Spec}\,k$. We will show that there exists a closed point $\tilde{x}$ of $\mathcal{X}$ such that for any vector bundle E on $\mathcal{X}$, $\Phi(E)$ is determined by the restriction of E to $\tilde{x}$. Let $p : \mathcal{X} \to M$ denote the coarse moduli map. Since $\Phi \circ p^*$ is the composite of an integral functor and p^*, and M satisfies the condition (i) in Section 2.3, thus by Corollary 4.7, Theorem 5.1 and Proposition 5.8 (see Remark 4.9) there exists a morphism $x : S = \mathrm{Spec}\,k \to M$ such that $x^* \simeq \Phi \circ p^*$. This morphism x determines a closed point of M which we denote by the same letter x. Let $\widehat{\mathcal{O}}_{M,x}$ be the completion of the local ring $\mathcal{O}_{M,x}$ and let $\widehat{\mathcal{O}}_{\mathcal{X},p^{-1}(x)}$ be the completion of $\mathcal{O}_{\mathcal{X}}$ with respect to the ideal I of the closed substack $p^{-1}(x)$. Since $\mathcal{O}_{M,x}$ is noetherian, $\widehat{\mathcal{O}}_{M,x}$ is a flat $\mathcal{O}_M$-module. Thus we have $x^*\widehat{\mathcal{O}}_{M,x} \simeq k$ and $p^*\widehat{\mathcal{O}}_{M,x} \simeq \widehat{\mathcal{O}}_{\mathcal{X},p^{-1}(x)}$. Therefore we have

$$
\Phi(E) \simeq \Phi(E) \otimes x^*\widehat{\mathcal{O}}_{M,x} \simeq \Phi(E) \otimes \Phi(p^*\widehat{\mathcal{O}}_{M,x}) \simeq \Phi(E \otimes p^*\widehat{\mathcal{O}}_{M,x}) \simeq \Phi(E \otimes \widehat{\mathcal{O}}_{\mathcal{X},p^{-1}(x)}).
$$

This means that $\Phi(E)$ is determined by the pullback of E to the stack $\mathcal{X}' := \mathcal{X} \times_M \operatorname{Spec} \widehat{\mathcal{O}}_{M,x}$. Since p is proper, by the Grothendieck's existence theorem for stacks [27, Theorem 1.4], the category of coherent sheaves on $\mathcal{X}'$ is equivalent to the category of compatible systems $\{(F_n', \phi_n' : F_{n+1}'/m^{n+1}F_{n+1}' \xrightarrow{\sim} F_n')_{n \geq 0}\}$ of coherent sheaves on the reductions $\mathcal{X}_n' := \mathcal{X} \times_M \operatorname{Spec}(\mathcal{O}_{M,x}/m^{n+1})$ where m is the maximal ideal of $\mathcal{O}_{M,x}$ and ϕ_n' is an isomorphism of coherent sheaves. Let $J \subset \mathcal{O}_{\mathcal{X}}$ be the ideal of the closed substack $(\mathcal{X}_0')_{\mathrm{red}}$ and $\mathcal{X}_n$ denote the closed substack defined by J^{n+1}. Then the category of compatible systems of coherent sheaves on $\mathcal{X}_n'$ is equivalent to the category of compatible systems of coherent sheaves on $\mathcal{X}_n$. Therefore we can regard any vector bundle E' on $\mathcal{X}'$ as a system $\{E_n\}_{n \geq 0}$ where E_n is a vector bundle on $\mathcal{X}_n$ and E_{n+1} is a flat deformation of E_n to $\mathcal{X}_{n+1}$. We will observe that this system $\{E_n\}_{n \geq 0}$ is determined by E_0. According to the deformation theory of modules over a ringed topos [10, IV, Proposition 3.1.5], the set of isomorphism classes of flat deformations of E_n to $\mathcal{X}_{n+1}$ is a torsor under $\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{X}_n}}(E_n, E_n \otimes_{\mathcal{O}_{\mathcal{X}_n}} J^{n+1}/J^{n+2})$ (note that Cartesian modules are stable under deformations) and we have

$$\begin{aligned}
\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{X}_n}}(E_n, E_n \otimes_{\mathcal{O}_{\mathcal{X}_n}} J^{n+1}/J^{n+2}) &\simeq \mathrm{H}^1(\mathcal{X}_n, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}_n}}(E_n, E_n \otimes_{\mathcal{O}_{\mathcal{X}_n}} J^{n+1}/J^{n+2})) \\
&\simeq \mathrm{H}^1(\mathcal{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}_0}}(E_0, E_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} J^{n+1}/J^{n+2})).
\end{aligned}$$

Let $\tilde{x} : \operatorname{Spec} k \to \mathcal{X}$ be a point of $\mathcal{X}$ such that $p \circ \tilde{x} = x$. Then $\mathcal{X}_0$ is isomorphic to the residual gerbe of $\tilde{x}$ over k and this gerbe is isomorphic to the classifying stack $BG_{\tilde{x}}$, where $G_{\tilde{x}}$ is the stabilizer group of $\tilde{x}$. Since $\mathcal{X}$ is tame, $G_{\tilde{x}}$ is linearly reductive and hence $\mathrm{H}^1(\mathcal{X}_0, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}_0}}(E_0, E_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} J^{n+1}/J^{n+2})) = 0$. Therefore the system $\{E_n\}$ is determined by E_0. In addition, in our setting $G_{\tilde{x}}$ is finite. Hence the number of finite dimensional irreducible representations of $G_{\tilde{x}}$ is finite and any representation is completely reducible. In other words, there exist vector bundles $E_{01}, \ldots, E_{0n}$ on $\mathcal{X}_0$ such that any vector bundle on $\mathcal{X}_0$ is isomorphic to a sheaf of the form $\bigoplus E_{0i}^{\oplus a_i}$. By the deformation theory and the Grothendieck's existence theorem, for any i, there exists an object F_i in $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ which is a locally free $\widehat{\mathcal{O}}_{\mathcal{X}, p^{-1}(x)}$-module and whose restriction to $\mathcal{X}_0$ is isomorphic to E_{0i}. Thus for any vector bundle E on $\mathcal{X}$, $\Phi(E)$ is quasi-isomorphic to a complex of the form $\bigoplus \Phi(F_i)^{\oplus a_i}$. If $a_i \neq 0$, the complex $\Phi(F_i)$ is bounded since $\Phi(E)$ is bounded for any vector bundle E. Hence the family $\Phi(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$ is uniformly bounded, i.e., there exist integers $a \leq b$ such that for any vector bundle E, $\mathrm{H}^l(\Phi(E)) = 0$ if $l \notin [a, b]$. Therefore, by Remark 4.4, it follows that Φ preserves vector bundles.

Case 2. We then consider the case that k is algebraically closed and S is a general affine scheme over k. We will prove that the family $\Phi(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$ is uniformly bounded above (Remark 4.4). If this family is not uniformly bounded above, there exist a vector bundle E on $\mathcal{X}$ such that the integer $m = \max\{l \mid \mathrm{H}^l(\Phi(E)) \neq 0\}$ is positive. Since $\mathrm{H}^m(\Phi(E))$ is finitely generated, by Nakayama's lemma, there exist a field K and a morphism $a : \operatorname{Spec} K \to S$ such that $\mathrm{H}^m(a^* \Phi(E))$ is not zero. Hence we may and will assume that $S = \operatorname{Spec} K$. By Corollary 4.7, Theorem 5.1 and Proposition 5.8 there exists a morphism $f : S \to M$ such that $f^* \simeq \Phi \circ p^*$. Since M is of finite type over k, there exist a k-subalgebra R of K of finite type and $g : T = \operatorname{Spec} R \to M$ such that $f = g \circ h$ where $h : S \to T$ is the morphism induced by the inclusion $R \subset K$. We have

obtained the following homotopy commutative diagram:

$$\begin{array}{ccc} \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) & \xrightarrow{\ \Phi\ } & \mathcal{D}_{\mathrm{qcoh}}(S = \operatorname{Spec} K) \\ {\scriptstyle p^*}\big\uparrow & {\scriptstyle f^*}\nearrow & \big\uparrow{\scriptstyle h^*} \\ \mathcal{D}_{\mathrm{qcoh}}(M) & \xrightarrow{\ g^*\ } & \mathcal{D}_{\mathrm{qcoh}}(T = \operatorname{Spec} R). \end{array}$$

Let $\xi : \operatorname{Spec} k \to T$ be a closed point of T and let x denote the closed point $g \circ \xi :$ $\operatorname{Spec} k \to M$. By the similar argument as in Case 1, there exist objects $F_1, \dots, F_n$ in $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ such that $\Phi(F_i)$ is bounded above and for any vector bundle E on $\mathcal{X}$, $E \otimes \widehat{\mathcal{O}}_{\mathcal{X}, p^{-1}(x)}$ is quasi-isomorphic to a complex of the form $\bigoplus_j F_j^{\oplus a_j}$. Hence $\Phi(E \otimes \widehat{\mathcal{O}}_{\mathcal{X}, p^{-1}(x)})$ is quasi-isomorphic to a complex of the form $\bigoplus_j \Phi(F_j)^{\oplus a_j}$ and the family $\{\Phi(E \otimes \widehat{\mathcal{O}}_{\mathcal{X}, p^{-1}(x)})\}_{E \in \mathcal{D}_{\mathrm{vect}}(\mathcal{X})}$ is uniformly bounded above. On the other hand, we have

$$\Phi(E \otimes \widehat{\mathcal{O}}_{\mathcal{X}, p^{-1}(x)}) \simeq \Phi(E) \otimes f^* \widehat{\mathcal{O}}_{M,x}$$

Note that $f^* \widehat{\mathcal{O}}_{M,x}$ is not zero. Hence the family $\Phi(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$ is uniformly bounded above.

Case 3. Here we consider the case of an arbitrary base field k. Let $k \subset \overline{k}$ be an algebraic closure. As in Case 2 we may and will assume that S is $\operatorname{Spec} K$ where K is a field. For an algebraic stack $\mathcal{Y}$ over k we will write $\overline{\mathcal{Y}}$ for $\mathcal{Y} \times_k \overline{k}$. By [4, Theorem 4.7.], $\mathcal{D}_{\mathrm{qcoh}}(\overline{\mathcal{X}})$ and $\mathcal{D}_{\mathrm{qcoh}}(\overline{S})$ are naturally equivalent to

$$\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \otimes_{\mathcal{D}_{\mathrm{qcoh}}(\operatorname{Spec} k)} \mathcal{D}_{\mathrm{qcoh}}(\operatorname{Spec} \overline{k}) \quad \text{and} \quad \mathcal{D}_{\mathrm{qcoh}}(S) \otimes_{\mathcal{D}_{\mathrm{qcoh}}(\operatorname{Spec} k)} \mathcal{D}_{\mathrm{qcoh}}(\operatorname{Spec} \overline{k})$$

respectively (see [4] for the notation). Thus we have $\overline{\Phi} = \Phi \otimes_{\mathcal{D}_{\mathrm{qcoh}}(k)} \mathcal{D}_{\mathrm{qcoh}}(\overline{k}) :$ $\mathcal{D}_{\mathrm{qcoh}}(\overline{\mathcal{X}}) \to \mathcal{D}_{\mathrm{qcoh}}(\overline{S})$. Let E be a vector bundle on $\mathcal{X}$. It suffices to show that $\Phi(E)$ is (quasi-isomorphic to) a locally free sheaf in $\mathcal{D}_{\mathrm{qcoh}}(\overline{S})$, that is, $\overline{\Phi}(E)$ is a locally free sheaf. To complete the proof, we will reduce this case to the Case 1 and 2. Let $f : S \to M$ be a morphism such that $\Phi \circ p^* \simeq f^*$. If $\overline{p}$ and $\overline{f}$ denote the base changes of the coarse moduli map $p : \mathcal{X} \to M$ and $f : S \to M$ respectively, then $\overline{\Phi} \circ \overline{p}^* \simeq \overline{f}^*$ since external products of objects in $\mathcal{D}_{\mathrm{qcoh}}(M)$ and $\mathcal{D}_{\mathrm{qcoh}}(\overline{k})$ generate $\mathcal{D}_{\mathrm{qcoh}}(\overline{M})$ (see [4, Section 4.2]). Let F be a quasi-coherent sheaf on $\overline{\mathcal{X}}$. Since F is an inductive limit of external products of objects in $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ and $\mathcal{D}_{\mathrm{qcoh}}(\overline{k})$, thus it follows that $\overline{\Phi}(E \otimes F) \simeq \overline{\Phi}(E) \otimes \overline{\Phi}(F)$. Consequently, we can apply the arguments in Case 1 and 2 to $\overline{\Phi}$ and complete the proof. $\square$

Remark 4.9. The proof of Proposition 4.8 uses Proposition 5.8 of the case (i) of Section 2.3. But the proof of Proposition 5.8 for the scheme case (i) does not need Proposition 4.8.

As a consequence of this section, we here record the following corollary which follows from Theorem 4.5, Corollary 4.7, Proposition 4.8 and Theorem 5.1. See Section 5 for the notion of ∞-categorical symmetric monoidal functor, but readers who are not familiar with it might skip to the next section.

Corollary 4.10. *Let $\mathcal{X}$ be an algebraic stack that satisfies either (i) or (ii) in Section 2.3. Let S be an affine scheme over k. Let $\Phi : \mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S)$ be a symmetric monoidal colimit-preserving functor over $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(k)$. Then Φ preserves vector bundles,*

and there exist a k-morphism $f : S \to \mathcal{X}$ and a monoidal natural transformation $f^*|_{\mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X})} \simeq \Phi|_{\mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X})}$.

5. Derived Tannaka duality

In this section using results of Section 3 and 4 we prove main results of this paper Theorem 5.9, Corollary 5.11 and Theorem 5.13.

We here use the theory of symmetric monoidal ∞-categories developed in [23]. We refer to [23] for its generalities. Let $\mathcal{F}\mathrm{in}_*$ be the category of marked finite sets (cf. [23, 1.1]). Namely, objects are marked finite sets and a morphism from $\langle n \rangle_* := \{1 < \cdots < n\} \sqcup \{*\} \to \langle m \rangle_* := \{1 < \cdots < m\} \sqcup \{*\}$ is a (not necessarily order-preserving) map of finite sets which preserves the distinguished points $*$. Let $\alpha^{i,n} : \langle n \rangle_* \to \langle 1 \rangle_*$ be a map such that $\alpha^{i,n}(i) = 1$ and $\alpha^{i,n}(j) = *$ if $i \neq j \in \langle n \rangle_*$. Let $I := \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. A symmetric monoidal category is a coCartesian fibration (cf. [18, 2.4]) $p : \mathcal{M}^{\otimes} \to I$ such that for any $n \geq 0$, $\alpha^{1,n} \ldots \alpha^{n,n}$ induce an equivalence $\mathcal{M}^{\otimes}_n \to (\mathcal{M}^{\otimes}_1)^{\times n}$ where $\mathcal{M}^{\otimes}_n$ and $\mathcal{M}^{\otimes}_1$ are fibers of p over $\langle n \rangle_*$ and $\langle 1 \rangle_*$ respectively. Let $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\Delta,\mathrm{sMon}}$ be the simplicial category of symmetric monoidal ∞-categories in which morphisms are symmetric monoidal functors (see [23, 1.18]). Let $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\mathrm{sMon}}$ be the simplicial nerve of $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\Delta,\mathrm{sMon}}$. For a symmetric monoidal ∞-category $\mathcal{M}$, we let $\widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{M}/}^{\mathrm{sMon}}$ the undercategory in the obvious manner. We shall refer to a morphism in $\widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k)/}^{\mathrm{sMon}}$ as a k-linear symmetric monoidal functor. For an affine k-scheme T, we denote by $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(T)$ the ∞-category $\mathcal{D}_{\mathrm{qcoh}}(T)$ endowed with the natural symmetric monoidal structure. Let $\mathcal{X}$ be an algebraic stack over a field k. Let J be the category of affine k-schemes over $\mathcal{X}$. Then using the construction [18, 8.3] and the simplicial nerve functor we have the functor $(J^{op})^{\lhd} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\Delta,\mathrm{sMon}}$ sending $T \to \mathcal{X} \in J$ to $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(T)$ and sending $f : T' \to T$ to $f^* : \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(T) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(T')$, and the cone point maps to $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k)$ by the structure maps $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(T)$ ($(\bullet)^{\lhd}$ indicates the left cone [18, 1.2.8]). Thus if we abusively write J for the nerve of J, we have $J^{op} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k)/}^{\mathrm{sMon}}$. Take a limit $(J^{op})^{\lhd} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k)/}^{\mathrm{sMon}}$ of the diagram and we shall denote the limit by $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X})$. The ∞-category $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ endows with a symmetric monoidal structure given by Cartesian product $\mathcal{C} \times \mathcal{D}$ [22, 4.1,6.2] and a symmetric monoidal ∞-category can be viewed as a commutative algebra object in $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$. Thus $\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\mathrm{sMon}}$ is equivalent to the ∞-category of commutative algebra objects in $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$. By applying [23, 4.17, 5.6, 5.9] and [22, 2.3.5] to $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$, the underlying category of $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X})$ is equivalent to $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$. If $\mathcal{N}, \mathcal{N}' \in \widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{M}/}^{\mathrm{sMon}}$ we shall denote by $\mathrm{Map}^{\otimes}_{\mathcal{M}}(\mathcal{N}, \mathcal{N}')$ the mapping space from $\mathcal{N}$ to $\mathcal{N}'$ in $\widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{M}/}^{\mathrm{sMon}}$. We usually write $\mathrm{Map}^{\otimes}_{\mathcal{D}(k)}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$ for $\mathrm{Map}^{\otimes}_{\mathcal{D}^{\otimes}(k)}(\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S))$. Let $\mathrm{Map}^{\otimes}_k(\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S))$ (or simply $\mathrm{Map}^{\otimes}_k(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$) denote the full subcategory of $\mathrm{Map}^{\otimes}_{\mathcal{D}(k)}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$, spanned by colimit-preserving functors. We can regard $\mathrm{Map}^{\otimes}_{\mathcal{D}(k)}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$ as the homotopy product of

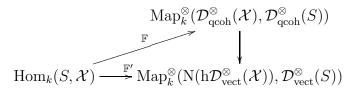$$\mathrm{Map}_{\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\Delta,\mathrm{sMon}}}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S)) \to \mathrm{Map}_{\widehat{\mathcal{C}\mathrm{at}}_{\infty}^{\Delta,\mathrm{sMon}}}(\mathcal{D}_{\mathrm{qcoh}}(k), \mathcal{D}_{\mathrm{qcoh}}(S)) \leftarrow \{*\},$$

where the first map is induced by the structure map $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X})$ and the second map is induced by the structure map $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S)$ (so using the projective model structure of the diagram category we have a functor $J^{op} \to \mathrm{Set}_{\Delta}, S \mapsto \mathrm{Map}^{\otimes}_{\mathcal{D}(k)}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S)))$. Let $\mathrm{Hom}_k(S, \mathcal{X})$ denote a 1-groupoid of k-morphisms from a fixed affine k-scheme S to an algebraic stack $\mathcal{X}$. We shall regard $\mathrm{Hom}_k(S, \mathcal{X})$ as the (simplicial) nerve of $\mathrm{Hom}_k(S, \mathcal{X})$. Then there is the natural map $\mathrm{Hom}_k(S, \mathcal{X})^{op} \to J^{op}$ and it extends to a map of left cones (cf. [18]): $(\mathrm{Hom}_k(S, \mathcal{X})^{op})^{\triangleleft} \to (J^{op})^{\triangleleft}$. By composing $(J^{op})^{\triangleleft} \longrightarrow \widehat{\mathcal{C}\mathrm{at}}^{\mathrm{sMon}}_{\infty, \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k)/}$ with $(\mathrm{Hom}_k(S, \mathcal{X})^{op})^{\triangleleft} \to (J^{op})^{\triangleleft}$ we have

$$\mathbb{F} : \mathrm{Hom}_k(S, \mathcal{X}) \to \mathrm{Map}^{\otimes}_k(\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S)),$$

where we here view the right-hand side as the Kan complex of left morphisms [18, 1.2.2.3] or Section 2).

Let $\mathcal{X}$ be an algebraic stack that satisfies either (i) or (ii) in Section 2.3. In virtue of Corollary 4.10 any k-linear symmetric monoidal colimit-preserving functor $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S)$ preserves vector bundles. Thus there is a diagram

$$
\begin{array}{ccc}
 & \mathrm{Map}^{\otimes}_k(\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S)) & \\
 & \nearrow^{\mathbb{F}} \qquad\qquad \downarrow & \\
\mathrm{Hom}_k(S, \mathcal{X}) \xrightarrow{\ \mathbb{F}'\ } & \mathrm{Map}^{\otimes}_k(\mathrm{N}(\mathrm{h}\mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X})), \mathcal{D}^{\otimes}_{\mathrm{vect}}(S)) &
\end{array}
$$

in $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$, where the vertical arrow is induced by the restriction, $\mathbb{F}'$ sends $f : S \to \mathcal{X}$ to the k-linear symmetric monoidal functor $f^* : \mathrm{N}(\mathrm{h}\mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X})) \to \mathcal{D}^{\otimes}_{\mathrm{vect}}(S)$, and $\mathrm{Map}^{\otimes}_k(\mathrm{N}(\mathrm{h}\mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X})), \mathcal{D}^{\otimes}_{\mathrm{vect}}(S))$ is the category of k-linear symmetric monoidal additive exact functors in which morphisms are monoidal natural transformations.

The following is Tannaka duality for quasi-projective schemes with action of an affine group scheme (generalizing the classical one), proved by Savin [30]. In [19], Lurie shows another version of Tannaka duality for a geometric stack using the symmetric monoidal category of quasi-coherent sheaves.

Theorem 5.1 ([30]). *Suppose that $\mathcal{X}$ is a quotient stack of the form $[X/G]$, where X is a separated noetherian scheme and G is a linear algebraic group acting on X. Suppose that there is a very ample G-invertible sheaf on X. The functor $\mathbb{F}'$ is a categorical equivalence.*

We will prepare some lemmas. According to the straightening [18, 3.2.0.1], a co-Cartesian fibration $p : \mathcal{M}^{\otimes} \to I$ amounts to a functor $I \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. More precisely, $\widehat{\mathcal{C}\mathrm{at}}^{\mathrm{sMon}}_{\infty}$ can be embedded into $\mathrm{Fun}(I, \widehat{\mathcal{C}\mathrm{at}}_{\infty})$ as a full subcategory (see [22, 1.2.15]). Let $\alpha, \beta, \gamma : I \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ be functors corresponding to $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X})$, $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S)$ and $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k)$ respectively. For instance, $\alpha : I \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ is depicted as

$$\cdots \rightrightarrows \mathcal{C}^i \rightrightarrows \mathcal{C}^{i-1} \rightrightarrows \cdots \rightrightarrows \mathcal{C}^2 \rightrightarrows \mathcal{C}^1 \rightrightarrows \mathcal{C}^0 \simeq *$$

where $\mathcal{C}^i$ is an ∞-category which is categorically equivalent to the i-times product of $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$, and $\rightrightarrows$ between $\mathcal{C}^i$ and $\mathcal{C}^{i-1}$ informally represents morphisms induced by maps between $\langle i \rangle_*$ and $\langle i-1 \rangle_*$ (namely, $\mathcal{C}^i$ is $\alpha(\langle i \rangle_*)$). Here we use notation similar to Section 3, i.e., $\mathcal{C} = \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$, $\mathcal{D} = \mathcal{D}_{\mathrm{qcoh}}(S)$, $\mathcal{K} = \mathcal{D}_{\mathrm{qcoh}}(k)$. The functors

$\beta, \gamma : I \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ are described in the similar way. A symmetric monoidal functor $\mathcal{D}^\otimes_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}^\otimes_{\mathrm{qcoh}}(S)$ amounts to a natural transformation $\alpha \to \beta$, that is, a morphism from α to β in $\mathrm{Fun}(I, \widehat{\mathcal{C}\mathrm{at}}_\infty)$. It is described by the diagram in $\widehat{\mathcal{C}\mathrm{at}}_\infty$:

$$
\begin{array}{ccccccccccc}
\cdots & \rightleftarrows & \mathcal{C}^i & \rightleftarrows & \mathcal{C}^{i-1} & \rightleftarrows & \cdots & \rightleftarrows & \mathcal{C}^2 & \rightleftarrows & \mathcal{C}^1 & \rightleftarrows & \mathcal{C}^0 \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \rightleftarrows & \mathcal{D}^i & \rightleftarrows & \mathcal{D}^{i-1} & \rightleftarrows & \cdots & \rightleftarrows & \mathcal{D}^2 & \rightleftarrows & \mathcal{D}^1 & \rightleftarrows & \mathcal{D}^0.
\end{array}
$$

We define a fibrant simplicial category $\mathscr{C}$. Objects of $\mathscr{C}$ are $\mathcal{C}^i, \mathcal{D}^i$ and $\mathcal{K}^i$ ($i \geq 0$). For simplicity, $\mathcal{C}^i = \mathcal{A}^i_1$, $\mathcal{D}^i = \mathcal{A}^i_2$, and $\mathcal{K}^i = \mathcal{A}^i_0$. For $0 \leq r, s \leq 2$ a simplicial set $\mathrm{Map}_\mathscr{C}(\mathcal{A}^i_r, \mathcal{A}^j_s)$ is $\mathrm{Map}(\mathcal{A}^i_r, \mathcal{A}^j_s)$.

Let $\mathcal{B}^i_r$ denote the full subcategory of $\mathcal{A}^i_r$, consisting of vector bundles, that is, objects belonging to $((\mathcal{A}_r)_{\mathrm{vect}})^{\times i}$. Here $(\mathcal{A}_r)_{\mathrm{vect}}$ denotes the full subcategory of $\mathcal{A}_r$ spanned by complexes which are quasi-isomorphic to vector bundles. Let K be a small simplicial set. Let $K \to \mathcal{A}^i_r$ be a functor which has the property: there is $c \in \{1, \ldots, i\}$ such that for $j \neq c$ the composite $K \to \mathcal{A}^1_r$ with the j-th natural projection $\alpha^{j,i}_! : \mathcal{A}^i_r \to \mathcal{A}^1_r$ is equivalent to a constant diagram. We will call such a functor a diagram of one-variable indexed by K. We say that a functor $\mathcal{A}^i_r \to \mathcal{A}^j_s$ is good if for any simplicial set K and a diagram of one-variable $K \to \mathcal{A}^i_r$ the composite $K \to \mathcal{A}^j_s$ is a diagram of one-variable.

Let us define a simplicial subcategory $\mathscr{P}$ of $\mathscr{C}$ as follows. A collection of objects of $\mathscr{P}$ is the same as that of $\mathscr{C}$. We define hom simplicial sets as follows: For any $0 \leq r, s \leq 2$, $\mathrm{Map}_\mathscr{P}(\mathcal{A}^i_r, \mathcal{A}^j_s)$ is the full subcategory of $\mathrm{Map}(\mathcal{A}^i_r, \mathcal{A}^j_s)$, spanned by good functors which are colimit-preserving separately in each variable of $\mathcal{A}^i_r$ and send $\mathcal{B}^j_r$ to $\mathcal{B}^j_s$. These data constitute a simplicial category $\mathscr{P}$.

Next we will define another fibrant simplicial category $\mathscr{Q}$. The collection of objects is the same as $\mathscr{P}$. For any $0 \leq r, s \leq 2$, $\mathrm{Map}_\mathscr{Q}(\mathcal{A}^i_r, \mathcal{A}^j_s)$ is the full subcategory of $\mathrm{Map}(\mathcal{B}^i_r, \mathcal{B}^j_s)$, spanned by functors which are equivalent to image of the restriction $\mathrm{Map}_\mathscr{P}(\mathcal{A}^i_r, \mathcal{A}^j_s) \to \mathrm{Map}(\mathcal{B}^i_r, \mathcal{B}^j_s)$. Compositions are well-defined and the data form a simplicial category. There is a natural simplicial functor $\xi : \mathscr{P} \to \mathscr{Q}$. Then by Lemma 3.7 and 3.11, we have the following:

Lemma 5.2. *Let $\mathcal{X}$ and S be perfect stacks over k. Suppose that $\mathcal{X}$ and S have cohomological dimension zero. Then the simplicial functor $\xi : \mathscr{P} \to \mathscr{Q}$ of fibrant simplicial categories is an equivalence.*

Since α, β and γ factor through $\mathrm{N}(\mathscr{P})$ in the obvious manner, we write $\alpha', \beta', \gamma' : I \to \mathrm{N}(\mathscr{P})$ for their factorizations. A symmetric monoidal functor $\mathcal{D}^\otimes_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}^\otimes_{\mathrm{qcoh}}(S)$ can be viewed as a natural transformation $\alpha' \to \beta'$. On the other hand, a natural transformation between $\mathrm{N}(\xi) \circ \alpha', \mathrm{N}(\xi) \circ \beta' : I \to \mathrm{N}(\mathscr{P}) \to \mathrm{N}(\mathscr{Q})$ is nothing but a symmetric monoidal functor between $\mathcal{D}^\otimes_{\mathrm{vect}}(\mathcal{X})$ and $\mathcal{D}^\otimes_{\mathrm{vect}}(S)$. By these observations we deduce the following:

Proposition 5.3. *Suppose that $\mathcal{X}$ has cohomological dimension zero (and S is affine). Let $\Phi, \Psi : \mathcal{D}^\otimes_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}^\otimes_{\mathrm{qcoh}}(S)$ be k-linear symmetric monoidal colimit-preserving functors. Let $\overline{\Phi} : \mathcal{D}^\otimes_{\mathrm{vect}}(\mathcal{X}) \to \mathcal{D}^\otimes_{\mathrm{vect}}(S)$ and $\overline{\Psi} : \mathcal{D}^\otimes_{\mathrm{vect}}(\mathcal{X}) \to \mathcal{D}^\otimes_{\mathrm{vect}}(S)$ be the restriction of Φ and Ψ. (By Corollary 4.10, Φ and Ψ preserve vector bundles.) Then the natural functor*

$$\mathrm{Map}(\Phi, \Psi) \to \mathrm{Map}(\overline{\Phi}, \overline{\Psi})$$

is a weak homotopy equivalence. Here we denote by $\mathrm{Map}(\Phi, \Psi)$ *and* $\mathrm{Map}(\overline{\Phi}, \overline{\Psi})$ *the mapping spaces in* $\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$ *and* $\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathcal{D}_{\mathrm{vect}}^{\otimes}(\mathcal{X}), \mathcal{D}_{\mathrm{vect}}^{\otimes}(S))$ *respectively.*

Remark 5.4. Using an argument which is similar to Proposition 5.3 and Lemma 3.7, we deduce that there is an natural equivalence

$$\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S)) \to \mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{perf}}(\mathcal{X}), \mathcal{D}_{\mathrm{perf}}(S)).$$

where $\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{perf}}^{\otimes}(\mathcal{X}), \mathcal{D}_{\mathrm{perf}}^{\otimes}(S))$ is the mapping space of symmetric monoidal exact functors $\mathcal{D}_{\mathrm{perf}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{perf}}(S)$ over $\mathcal{D}_{\mathrm{perf}}^{\otimes}(k)$. (Note that symmetric monoidal functors preserve dualizable objects, i.e., perfect complexes.) It also follows from [23, Section 6]. Thus we can replace $\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S))$ by $\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{perf}}^{\otimes}(\mathcal{X}), \mathcal{D}_{\mathrm{perf}}^{\otimes}(S))$ in Theorem 5.9.

Lemma 5.5. *The restriction functor* $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{U})$ *induces a categorical equivalence*

$$\mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{U}), \mathcal{D}_{\mathrm{qcoh}}(S)) \to \mathrm{Map}_{\diamond}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S)).$$

The notation $\mathrm{Map}_{\diamond}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$ *indicates the full subcategory spanned by functors* Φ *such that if* $f : H \to G \in \mathrm{Fun}(\Delta^1, \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}))$ *induces an equivalence in* $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{U})$ *then* $\Phi(f)$ *is an equivalence.*

Proof. Let $p : \mathcal{C}^{\otimes} \to I$ and $q : \mathcal{D}^{\otimes} \to I$ be coCartesian fibrations that correspond to the symmetric monoidal ∞-categories $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X})$ and $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S)$. Let $\mathcal{U} \subset \mathcal{X}$ be an open substack. Let $\mathcal{C}_{\mathcal{U}}^{\otimes} \subset \mathcal{C}^{\otimes}$ be the full subcategory such that $\mathcal{C}_{\mathcal{U}}^{\otimes} \cap p^{-1}(\langle n \rangle_*)$ is spanned by $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{U}) \times \cdots \times \mathcal{D}_{\mathrm{qcoh}}(\mathcal{U})$ (n-times product). Namely, $p_{\mathcal{U}} : \mathcal{C}_{\mathcal{U}}^{\otimes} \to I$ is a coCartesian fibration that corresponds to $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{U})$. Let $\Phi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor such that if $f : H \to G \in \mathrm{Fun}(\Delta^1, \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}))$ induces an equivalence in $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{U})$ then $\Phi(f)$ is an equivalence. To prove our claim, in the light of [18, 4.3.2.15] and Lemma 3.12 it will suffice to show that $\Phi(P)$ is a q-limit of the diagram $(\mathcal{C}_{\mathcal{U}}^{\otimes})_{P/} \to \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ for any $P \in \mathcal{C}^{\otimes}$ where $(\mathcal{C}_{\mathcal{U}}^{\otimes})_{P/}$ denotes the undercategory. Suppose that $P \in p^{-1}(\langle n \rangle_*)$ and $P = [P_1, \ldots, P_n] \in \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})^{\times n}$. Let $P_{\mathcal{U}}$ be $[L(P_1), \ldots, L(P_n)] \in \mathcal{C}_{\mathcal{U}}^{\otimes}$ where $L : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{U}) \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ (cf. Lemma 3.12). Here we refer to $P_{\mathcal{U}}$ as a $\mathcal{U}$-localization of P. Since a $\mathcal{U}$-localization of $\iota_!(P_{\mathcal{U}})$ is equivalent to a $\mathcal{U}$-localization of $\iota_!(P)$ for any $\iota \in \mathrm{Fun}(\Delta^1, I)$ and p is a coCartesian fibration, we see that $P \to P_{\mathcal{U}}$ is an initial object of $\mathcal{C}_{\mathcal{U}}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{C}_{P/}^{\otimes}$ (cf. [18, 5.2.7.6]). Thus unwinding the definition of q-limits [18, 4.3.1.1] we conclude that $\Phi(P)$ is a q-limit of $(\mathcal{C}_{\mathcal{U}}^{\otimes})_{P/} \to \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$.   □

Let $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(k) \to \mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{U})$ and $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(k) \to \mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X})$ be k-linear structure maps. These maps induce $u : \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{U}), \mathcal{D}_{\mathrm{qcoh}}(S)) \to \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(k), \mathcal{D}_{\mathrm{qcoh}}(S))$ and $v : \mathrm{Map}_{\diamond}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S)) \to \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(k), \mathcal{D}_{\mathrm{qcoh}}(S))$. Let $\iota : \Delta^0 = * \to \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(k), \mathcal{D}_{\mathrm{qcoh}}(S))$ be the map corresponds to $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(k) \to \mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S)$. Note that $\mathrm{Map}_{\mathcal{D}(k)}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{U}), \mathcal{D}_{\mathrm{qcoh}}(S))$ is a homotopy pullback of u along ι. Let us denote by $\mathrm{Map}_{\mathcal{D}(k), \diamond}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$ a homotopy pullback of v along ι.

Corollary 5.6. *There is a natural equivalence*

$$\mathrm{Map}_{\mathcal{D}(k)}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{U}), \mathcal{D}_{\mathrm{qcoh}}(S)) \to \mathrm{Map}_{\mathcal{D}(k), \diamond}^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S)).$$

Let $\mathcal{E} = \mathcal{D}_{\mathrm{vect}}^{\otimes}(\mathcal{X})$ be the symmetric monoidal ∞-subcategory. Let $\mathrm{E}^{\otimes} = \mathrm{N}(\mathrm{h}(\mathcal{E}^{\otimes}))$ be the homotopy category E which has the symmetric monoidal structure. Let $h : \mathcal{E}^{\otimes} \to \mathrm{E}^{\otimes}$ be the natural functor. Using the straightening functor [18, 3.2.0.1] we have functors

$$a, b, c : \mathcal{F}\mathrm{in}_* \to \widehat{\mathrm{Set}}_{\Delta}^{+}$$

corresponding to $\mathcal{E}^{\otimes}$, $\mathrm{E}^{\otimes}$ and $\mathcal{F}^{\otimes} := \mathcal{D}_{\mathrm{vect}}^{\otimes}(S)$ respectively, and the diagram

$$a \longrightarrow b \qquad c$$

where a, b and c are cofibrant-fibrant objects with respect to the projective model structure and $a \to b$ represents h. Next let us define a simplicial category $\mathscr{S}$ as follows. A collection of objects of $\mathscr{S}$ is $a(i), b(i), c(i)$ where $i \in \mathcal{F}\mathrm{in}_*$. By the projective model structure, we may assume that $a(i), b(i)$ and $c(i)$ are ∞-categories which are equivalent to $\mathcal{E}^{\times i}$, $\mathrm{E}^{\times i}$ and $\mathcal{F}^{\times i}$ respectively. Hom simplicial sets are defined as follows. Let $i, j \in \mathcal{F}\mathrm{in}_*$. Let $\mathrm{Map}_{\mathscr{S}}(c(i), c(j)) = \mathrm{Map}(c(i), c(j))$. For $x, y \in \{a, b\}$ (possibly $x = y$), a complex $\mathrm{Map}_{\mathscr{S}}(x(i), y(j))$ is a constant subcomplex of $\mathrm{Map}(x(i), y(j))$ consisting of the image of $a \to b$, i.e., functors of the form $x(i) \to y(i) \to y(j)$ induced by $a \to b$ and $i \to j \in \mathcal{F}\mathrm{in}_*$. (Note that in our situation if $x \in \{a, b\}$ and $f, g \in \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(i, j)$ and $f \neq g$, then two functors $x(i) \to x(j)$ and $x(i) \to x(j)$ induced by f and g respectively, are not equivalent to one another.) Let $\mathrm{Map}_{\mathscr{S}}(b(i), c(j)) = \mathrm{Map}(b(i), c(j))$ and $\mathrm{Map}_{\mathscr{S}}(a(i), c(j)) = \mathrm{Map}(a(i), c(j))$. If otherwise, hom simplicial sets are empty sets. Compositions are defined in the obvious manner and $\mathscr{S}$ forms a simplicial category. Let $\mathscr{S}'$ (resp. $\mathscr{S}''$) be a simplicial subcategory obtained by removing b (resp. a) from $\mathscr{S}$. We have a simplicial functor $l : \mathscr{S}'' \to \mathscr{S}'$ determined by

- $l(b(i)) = a(i)$
- $b(i) \to c(j)$ maps to the composite $a(i) \to b(i) \to c(j)$, that is, $\mathrm{Map}(b(i), c(j)) \to \mathrm{Map}(a(i), c(j))$ induced by $a(i) \to b(i)$.
- $b(i) \to b(j)$ maps to $a(i) \to a(j)$ for each $i \to j$
- if otherwise l sends objects and morphisms identically.

Suppose that S is affine. According to Lemma 3.1 the functor $a(i) \to b(i)$ induces a categorical equivalence $\mathrm{Map}(b(i), c(j)) \to \mathrm{Map}(a(i), c(j))$ (note that $\mathrm{h}(\mathcal{C}^{\times n})$ is equivalent to $(\mathrm{h}\mathcal{C})^{\times n}$). Thus $\mathscr{S}'' \to \mathscr{S}'$ is an equivalence of fibrant simplicial categories. Then $\mathrm{Map}_{\mathrm{Fun}(I, \mathrm{N}(\mathscr{S}''))}(\mathrm{N}(b), \mathrm{N}(c))$ is the space of symmetric monoidal functors $\mathrm{E}^{\otimes} \to \mathcal{F}^{\otimes}$. Similarly, $\mathrm{Map}_{\mathrm{Fun}(I, \mathrm{N}(\mathscr{S}'))}(\mathrm{N}(a), \mathrm{N}(c))$ is the space of symmetric monoidal functors $\mathcal{E}^{\otimes} \to \mathcal{F}^{\otimes}$. By the categorical equivalence $\mathrm{N}(\mathscr{S}'') \to \mathrm{N}(\mathscr{S}')$ we have:

Lemma 5.7. *Suppose that S is affine. Then the functor $\mathcal{D}_{\mathrm{vect}}^{\otimes}(\mathcal{X}) \to \mathrm{N}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}^{\otimes}(\mathcal{X}))$ induces a categorical equivalence*

$$\mathrm{Map}^{\otimes}(\mathrm{N}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X})), \mathcal{D}_{\mathrm{vect}}(S)) \to \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}), \mathcal{D}_{\mathrm{vect}}(S))$$

Moreover the equivalence induces an equivalence

$$\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{N}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X})), \mathcal{D}_{\mathrm{vect}}(S)) \to \mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}), \mathcal{D}_{\mathrm{vect}}(S)).$$

The latter follows from the fact: $\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{N}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X})), \mathcal{D}_{\mathrm{vect}}(S))$ is a homotopy pullback of $\mathrm{Map}^{\otimes}(\mathrm{N}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X})), \mathcal{D}_{\mathrm{vect}}(S)) \to \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(k), \mathcal{D}_{\mathrm{vect}}(S))$ (induced by $\mathcal{D}_{\mathrm{vect}}^{\otimes}(k) \to \mathrm{N}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$) along $* \to \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(k), \mathcal{D}_{\mathrm{vect}}(S))$ induced by the

k-linear structure $\mathcal{D}^{\otimes}_{\mathrm{vect}}(k) \to \mathcal{D}^{\otimes}_{\mathrm{vect}}(S)$. Similarly, $\mathrm{Map}^{\otimes}_{\mathcal{D}^{\otimes}_{\mathrm{vect}}(k)}(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}), \mathcal{D}_{\mathrm{vect}}(S))$ is a homotopy pullback of $\mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}), \mathcal{D}_{\mathrm{vect}}(S)) \to \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(k), \mathcal{D}_{\mathrm{vect}}(S))$ along $* \to \mathrm{Map}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(k), \mathcal{D}_{\mathrm{vect}}(S))$.

Proposition 5.8. *Let $\mathcal{X}$ be an algebraic stack over k that satisfies either (i) or (ii) in Section 2.3. Let S be an affine scheme over k. Let $a, b : \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S)$ be k-linear symmetric monoidal colimit-preserving functors (such that $a(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$ and $b(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}))$ lie in $\mathcal{D}_{\mathrm{vect}}(S)$). Let $\bar{a} := a|_{\mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X})} : \mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X}) \to \mathcal{D}^{\otimes}_{\mathrm{vect}}(S)$ and $\bar{b} := b|_{\mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X})} : \mathcal{D}^{\otimes}_{\mathrm{vect}}(\mathcal{X}) \to \mathcal{D}^{\otimes}_{\mathrm{vect}}(S)$. Suppose that there exist $x, y : S \rightrightarrows \mathcal{X}$ such that $\bar{a}$ and $\bar{b}$ are equivalent to pullback functors $x^* : \mathcal{D}_{\mathrm{vect}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{vect}}(S)$ and $y^* : \mathcal{D}_{\mathrm{vect}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{vect}}(S)$ as functors respectively. Then the restriction $\mathrm{Map}^{\otimes}(a, b) \to \mathrm{Map}^{\otimes}(\bar{a}, \bar{b})$ is a weak homotopy equivalence. Here $\mathrm{Map}^{\otimes}(a, b)$ is the mapping space from a to b in $\mathrm{Map}^{\otimes}_k(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$, and $\mathrm{Map}(\bar{a}, \bar{b})$ is the mapping space from $\bar{a}$ to $\bar{b}$ in $\mathrm{Map}^{\otimes}_{\mathcal{D}^{\otimes}_{\mathrm{vect}}(k)}(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}), \mathcal{D}_{\mathrm{vect}}(S))$.*

Proof. We first fix some notation. Take a Zariski affine covering $\sqcup_l S_l \to S$ such that each $S_l \to S \xrightarrow{x} \mathcal{X}$ (and $S_l \to S \xrightarrow{y} \mathcal{X}$) factors through a quasi-compact open sub-stack $\mathcal{U}_l \subset \mathcal{X}$ which has cohomological dimension zero. Let Z be the category of affine schemes T over S such that $T \to S$ is an open immersion and $T \to S$ factors through some $S_l \subset S$. Then $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S)$ is a limit of the diagram $Z^{op} \to \widehat{\mathcal{C}\mathrm{at}}^{\mathrm{sMon}}_{\infty, \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(k)/}$ sending $T \in Z$ to $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(T)$. If $z \in Z$ indicates $T \to S$, then we write S^z for T and we denote by a^z the composite $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}) \xrightarrow{a} \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S^z)$, and we use the notation b^z and $\bar{a}^z$ in a similar manner. Note that by Lemma 3.2 and Remark 3.3 a bounded complex P on $\mathcal{X}$ which in each degree is an infinite direct sum of vector bundles is a finite colimit of infinite direct sums of vector bundles up to shifts. Thus $a(P) \simeq x^*(P)$ and $b(P) \simeq y^*(P)$. If $P \in \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ is acyclic on $\mathcal{U}_l$, then by Lemma 3.4 we conclude that $a(P)$ and $b(P)$ are acyclic on S_l. Thus by Lemma 5.5 each $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}) \xrightarrow{a} \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S_l)$ and $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}) \xrightarrow{b} \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(S_l)$ factor through $\mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}^{\otimes}_{\mathrm{qcoh}}(\mathcal{U}_l)$. To prove our claim, it is convenient to recall the presentation of the mapping spaces in an ∞-category, introduced in [18, page 28]. Let $\mathcal{C}$ be a Kan complex (for the case of ∞-categories see [18]) and let c and c' be two objects in $\mathcal{C}$, i.e., two vertices. We define a mapping Kan complex $\mathrm{Fun}^{(c,c')}(\Delta^1, \mathcal{C})$ to be the fiber product of $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\partial\Delta^1, \mathcal{C}) \leftarrow * = \{(c, c')\}$. Since $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\partial\Delta^1, \mathcal{C})$ is a Kan fibration induced by inclusion $\partial\Delta^1 \to \Delta^1$, the fiber product is a homotopy fiber product. Using this presentation and the universality of limits together with Lemma 5.5 and Remark 5.4, we have the following categorical equivalences

$\mathrm{Fun}^{(a,b)}(\Delta^1, \mathrm{Map}^{\otimes}_{\mathcal{D}(k)}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S)))$

$$\simeq \mathrm{Fun}^{(a,b)}(\Delta^1, \lim_{z \in Z} \mathrm{Map}^{\otimes}_{\mathcal{D}(k)}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S^z)))$$

$$\simeq \lim_{z \in Z} \mathrm{Fun}^{(a^z, b^z)}(\Delta^1, \mathrm{Map}^{\otimes}_{\mathcal{D}(k)}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S^z))$$

$$\simeq \lim_{z \in Z} \mathrm{Fun}^{(a^z, b^z)}(\Delta^1, \mathrm{Map}^{\otimes}_{\mathcal{D}(k)}(\mathcal{D}_{\mathrm{qcoh}}(\sqcup_l \mathcal{U}_l), \mathcal{D}_{\mathrm{qcoh}}(S^z)))$$

$$\simeq \lim_{z \in Z} \mathrm{Fun}^{((a')^z, (b')^z)}(\Delta^1, \mathrm{Map}^{\otimes}_k(\mathcal{D}_{\mathrm{perf}}(\sqcup_l \mathcal{U}_l), \mathcal{D}_{\mathrm{perf}}(S^z))).$$

Here $(a')^z$ and $(b')^z$ is the restriction of a^z and b^z to $\mathcal{D}_{\mathrm{perf}}^{\otimes}(\mathcal{X})$ respectively. Applying Proposition 5.3 we have

$$\lim_{z\in Z}\mathrm{Fun}^{((a')^z,(b')^z)}(\Delta^1,\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{perf}}(\sqcup_l\,\mathcal{U}_l),\mathcal{D}_{\mathrm{perf}}(S^z)))$$

$$\simeq \lim_{z\in Z}\mathrm{Fun}^{(\bar{a}^z,\bar{b}^z)}(\Delta^1,\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(\sqcup_l\,\mathcal{U}_l),\mathcal{D}_{\mathrm{vect}}(S^z)))$$

$$\simeq \lim_{z\in Z}\mathrm{Fun}^{(\bar{a}^z,\bar{b}^z)}(\Delta^1,\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\sqcup_l\,\mathcal{U}_l),\mathcal{D}_{\mathrm{vect}}(S^z))).$$

We abusively write $\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\bullet)$ for $\mathrm{N}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\bullet))$. Note that by Theorem 5.1 for a Zariski open substack $\mathcal{U}\subset\mathcal{X}$ the full subcategory of $\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{U}),\mathcal{D}_{\mathrm{vect}}(S^z))$, spanned by additive exact functors can be viewed as $\mathrm{Hom}_k(S^z,\mathcal{U})$. Thus the full subcategory of $\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{U}),\mathcal{D}_{\mathrm{vect}}(S^z))$ can be naturally viewed as a full subcategory of $\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X}),\mathcal{D}_{\mathrm{vect}}(S^z))$. Thus by these obsevations, the descent theory of vector bundles and Lemma 5.7 we have equivalences

$$\lim_{z\in Z}\mathrm{Fun}^{(\bar{a}^z,\bar{b}^z)}(\Delta^1,\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\sqcup_l\,\mathcal{U}_l),\mathcal{D}_{\mathrm{vect}}(S^z)))$$

$$\simeq \lim_{z\in Z}\mathrm{Fun}^{(\bar{a}^z,\bar{b}^z)}(\Delta^1,\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X}),\mathcal{D}_{\mathrm{vect}}(S^z)))$$

$$\simeq \mathrm{Fun}^{(\bar{a},\bar{b})}(\Delta^1,\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathrm{h}\mathcal{D}_{\mathrm{vect}}(\mathcal{X}),\mathcal{D}_{\mathrm{vect}}(S)))$$

$$\simeq \mathrm{Fun}^{(\bar{a},\bar{b})}(\Delta^1,\mathrm{Map}_{\mathcal{D}_{\mathrm{vect}}^{\otimes}(k)}^{\otimes}(\mathcal{D}_{\mathrm{vect}}(\mathcal{X}),\mathcal{D}_{\mathrm{vect}}(S))).$$

Therefore we obtain the desired equivalence.   □

Finally, we obtain our main goal:

Theorem 5.9. *Let $\mathcal{X}$ be an algebraic stack which satisfies either (i) or (ii) in Section 2.3. Let S be a scheme over k (we always assume that S is quasi-compact and has affine diagonal). Then there is a categorical equivalence*

$$\mathbb{F}:\mathrm{Hom}_k(S,\mathcal{X})\longrightarrow \mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}),\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S))$$

which sends $f:S\to\mathcal{X}$ to f^*.

Proof. If S is affine, our claim follows from Corollary 4.10, Proposition 5.8 and Theorem 5.1. If S is a scheme, take a Zariski covering $\sqcup T\to S$ by affine schemes. It gives rise to a simplicial scheme $S_\bullet\to S$. Then $\mathrm{Hom}_k(S,\mathcal{X})$ is a limit of the cosimplicial diagram of $\mathrm{Hom}_k(S_i,\mathcal{X})$ indexed by $i\in\Delta$. On the other hand, $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S)$ is a limit of the cosimplicial diagram $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S_i)$ in $\widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(k)/}^{\mathrm{sMon}}$. Since $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})\to\mathcal{D}_{\mathrm{qcoh}}(S)$ is colimit-preserving if and only if the composite $\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})\to\mathcal{D}_{\mathrm{qcoh}}(\sqcup T)$ is colimit-preserving, thus $\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}),\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S))$ is a limit of the cosimplicial diagram of $\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}),\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S_i))$. Now our assertion follows from the case where S is affine.
                                        □

Remark 5.10. We would like to explain the reason why we should employ the theory of $(\infty,1)$-categories. Note that morphisms to $\mathcal{X}$ have the descent property. Namely, if $p:S'\to S$ is an étale surjective morphism and $\mathrm{pr}_1,\mathrm{pr}_2:S'\times_S S'\rightrightarrows S'$ are the first and second projections, then a morphism $f':S'\to\mathcal{X}$ such that $\mathrm{pr}_1\circ f'=\mathrm{pr}_2\circ f'$ descents to a unique morphism $f:S\to\mathcal{X}$ such that $p\circ f=f'$. Now suppose that Tannaka

duality formulated with the triangulated categories holds. Then the descent property of morphisms to $\mathcal{X}$ implies that functors $\mathrm{D}(\mathcal{X}) \to \mathrm{D}(S)$ of triangulated categories of a certain type have the descent property, where $\mathrm{D}(\bullet)$ denotes the triangulated category of quasi-coherent complexes (or perfect complexes). However, we can not hope that the derived categories have a reasonable descent theory. One of sources of this problem comes from the fact that triangulated categories forget the structure of homotopy coherence which naturally arise from (co)chain complexes. Inspired by the derived algebraic geometry [36], [37], [20] and derived Morita theory [34], [4], in order to establish our Tannaka duality we use not triangulated categories but "enhanced higher categories" such as stable (symmetric monoidal) ∞-categories. The idea of usage of higher category theory could be found in algebraic K-theory [38].

We here call Theorem 5.9 *derived Tannaka duality*, which is a title of this section. But perhaps it is more appropriate to say that Theorem 5.9 is a *stable* analogue of Tannaka duality, although the term "stable analogue" is ambiguous as well as the term "derived analogue".

Let us consider the $(\infty\text{-})$stack on the étale site (Aff_k) of affine k-schemes:

$$\mathcal{F}_{\mathcal{X}} : (\mathrm{Aff}_k)^{op} \longrightarrow \mathcal{S}$$

which sends S to $\mathrm{Map}_k^{\otimes}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S))$. Here $\mathcal{S}$ is ∞-category of spaces (Kan complexes) [18, 1.2.16] and we view $\mathcal{F}_{\mathcal{X}}$ as an object in $\mathrm{Fun}((\mathrm{Aff}_k)^{op}, \mathcal{S})$ or the localization of $\mathrm{Fun}((\mathrm{Aff}_k)^{op}, \mathcal{S})$ with respect to the étale topology of (Aff_k) [18, 6.2.2]. An immediate consequence of Theorem 5.9 is:

Corollary 5.11. *Let $\mathcal{X}$ be an algebraic stack over k that satisfies the condition either (i) or (ii). Then the stack $\mathcal{X}$ over (Aff_k) is equivalent to $\mathcal{F}_{\mathcal{X}}$.*

Remark 5.12. The above corollary is a reconstruction result. Our reconstruction is of different nature from one in [1]. The point is that (i) in loc. cit., schemes are reconstructed as ringed spaces, whereas we reconstruct them as sheaves on (Aff_k) (so it is applicable to the case of stacks), (ii) on one hand we recover a scheme X from a symmetric monoidal ∞-category $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(X)$ or $\mathcal{D}_{\mathrm{perf}}^{\otimes}(X)$; on the other hand, in loc. cit., a scheme is recovered from a symmetric monoidal triangulated category $\mathrm{D}_{\mathrm{perf}}^{\otimes}(X)$. We expect that an enhancement of a symmetric monoidal triangulated category $\mathrm{D}_{\mathrm{qcoh}}(X)$ is unique in an appropriate sense. In this direction, in the recent paper [17] by Lunts and Orlov it is shown that for a quasi-projective variety X an dg-enhancement of a triangulated category $\mathrm{D}_{\mathrm{qcoh}}(X)$ is unique.

Let $f : S \to \mathcal{X}$ be a morphism of stacks. Then we have an adjoint pair

$$f^* : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \rightleftarrows \mathcal{D}_{\mathrm{qcoh}}(S) : f_*.$$

Conversely, when does an adjoint pair arise in such a way? The following is a categorical characterization of functors associated to morphisms $S \to \mathcal{X}$, that is, Theorem 5.9 implies a tannakian characterization theorem.

Theorem 5.13. *Let $\mathcal{X}$ be an algebraic stack over k, that satisfies either condition (i) or (ii) in Section 2.3. Let S be a scheme over k. Let $\Phi : \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}(S)$ be a colimit-preserving functor. Then there exists a morphism $f : S \to \mathcal{X}$ over k such that f^* is equivalent to Φ if and only if Φ is equivalent to a k-linear symmetric monoidal functor (as objects in $\mathrm{Map}(\mathcal{D}_{\mathrm{qcoh}}(\mathcal{X}), \mathcal{D}_{\mathrm{qcoh}}(S)))$.*

Corollary 5.14. *Let $\Psi : \mathcal{D}_{\mathrm{qcoh}}(S) \to \mathcal{D}_{\mathrm{qcoh}}(\mathcal{X})$ be a right adjoint functor i.e., an accessible and limit-preserving functor (see the ∞-categorical adjoint functor theorem [18, 5.5.2.9]). Under the the same assumption as Theorem 5.13, there is a k-morphism $f : S \to \mathcal{X}$ such that Ψ is equivalent to f_* if and only if a left adjoint Φ of Ψ is equivalent to the underlying functor of some k-linear symmetric monoidal functor $\mathcal{D}_{\mathrm{qcoh}}^{\otimes}(\mathcal{X}) \to \mathcal{D}_{\mathrm{qcoh}}^{\otimes}(S)$.*

Remark 5.15. The above characterization gives an answer to the question: "what is the relationship between the group of automorphisms of the derived category of a projective variety and the group of isomorphisms of the variety?" (see [3, Preface]). It is perhaps worth remarking that Corollary 5.13 is new even in the case $\mathcal{X}$ is a scheme as well as the main theorem.

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